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A revisit to the plane problem for low-frequency acoustic scattering by an elastic cylindrical shell

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Abstract
The proposed revisit to a classical problem in fluid–structure interaction is due to an interest in the analysis of the narrow resonances corresponding to a low-frequency fluid-borne wave, inspired by modeling and design of metamaterials. In this case, numerical implementations would greatly benefit from preliminary asymptotic predictions. The normal incidence of an acoustic wave is studied for a circular cylindrical shell governed by plane strain equations in elasticity. A novel high-order asymptotic procedure is established considering for the first time all the peculiarities of the low-frequency behavior of a thin fluid-loaded cylinder. The obtained results are exposed in the form suggested by the Resonance Scattering Theory. It is shown that the pressure scattered by rigid cylinder is the best choice for a background component. Simple explicit formulae for resonant frequencies, amplitudes, and widths are presented. They support various important observations, including comparison between widths and the error of the asymptotic expansion for frequencies.

Keywords
Thin elastic shell, low-frequency, acoustic, scattering, resonance, metamaterials

1. Introduction

Acoustic wave scattering by a thin elastic cylindrical shell is a classical problem in the field of underwater acoustics (e.g., see [1–4]). It has an exact solution in Fourier series involving Bessel functions (e.g., see [5]),

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which, however, is not always ideally suited not only for analytical inspection but also for numerical implementation. The major challenge consists of the evaluation of very narrow low-frequency scattering resonances (see [6,7]), which can easily be missed in computations without a preliminary asymptotic insight.

At the same time, asymptotic treatment of the problem in question is mainly restricted to higher frequencies, when the effect of shell curvature is not that crucial and reduced plate-like models may be considered (see [2,8]) and references therein. Moreover, such models are not originally oriented to coupled problems in fluid–structure interaction formally adopting the results earlier obtained for plates under prescribed mechanical loads. The observations above strongly motivate a revisit to a well-known formulation.

It is worth noting that the general asymptotic procedure underlying the three-dimensional (3D) to two-dimensional (2D) dimension reduction for thin elastic shells [2,9] fails for low-frequency vibrations of a hollow cylinder, since its mid-surface virtually does not experience extension and shear (see also [10,11]). In this case, an appropriate framework has been established only not long ago (see [12,13]).

We also remark that the asymptotic developments in the cited papers [12,13] considering a shell with traction-free faces cannot be readily extended to a fluid-loaded shell. In addition to what has been already mentioned, we reiterate that the effect of fluid results in more sophisticated boundary conditions along a fluid–solid interface than Neumann ones corresponding to a shell not contacting with fluid. The former support the so-called fluid-borne wave [6], which has recently been studied in detail for low-frequency bending of a fluid-loaded plate in Kaplunov et al. [14]. In particular, it has been shown that at leading order this wave is supported by plate stiffness and fluid mass; in doing so, the plate mass appears only at higher order. Thus, it is a genuine fluid-borne phenomenon.

Last several years, there is a fresh interest in the low-frequency domain, inspired by metamaterial modeling, including the analysis of most useful band gaps for periodic arrays of thin cylinders immersed into acoustic media (see [15–17]). This seems to be another motivation for revisiting a famous problem in fluid–structure interaction.

In this paper, we consider resonance scattering at normal incidence. The shell is governed by plain strain equations in linear isotropic elasticity. Several low-frequency circumferential modes are studied. A high-order asymptotic scheme is developed. It is mostly in line with that for a free shell [13] but operates with a scaling specific for a fluid-borne wave, analogous to the above-mentioned problem for a fluid-loaded flat plate [14].

The obtained explicit formulae are interpreted using the Resonance Scattering Theory [1,3], including simple expressions for resonance frequencies, widths, and amplitudes. Both the transverse shell displacement and scattered fluid pressure are analyzed. Numerical examples are presented for a thin aluminum shell immersed in water. It is also remarkable that the error of the refined two-term expansion for frequencies is still greater than the widths of related resonances. Nevertheless, the derived approximations provide a better idea of low-frequency scattering by a cylindrical shell.

2. Statement of the problem

Consider a plane problem for a circular cylindrical shell of thickness $2h$ with mid-surface radius R immersed in a compressible fluid. The shell is assumed to be thin, i.e., $\eta = h/R \ll 1$ is a small geometric parameter. Let the domain occupied by the shell be given by $0 \leq \theta < 2\pi$, $-1 \leq \zeta \leq 1$, where θ and ζ are the scaled orthogonal coordinates along the mid-surface of the shell, and ζ is the transverse coordinate taking the values $\zeta \geq 1$ outside the shell (see Figure 1).

The 2D equations in linear elasticity governing time-harmonic shell motion (below, the factor $\exp(i\omega t)$, where ω is the angular frequency and t is time, is omitted) are given by, see Kaplunov et al. [2]:

$$\frac{\partial \sigma_{32}}{\partial \zeta} + \frac{\eta}{1+\eta\zeta}\frac{\partial \sigma_{22}}{\partial \theta} + \frac{2\eta}{1+\eta\zeta}\sigma_{32} + \eta\rho\omega^2 R v_2 = 0, \tag{1}$$

and

$$\frac{\partial \sigma_{33}}{\partial \zeta} + \frac{\eta}{1+\eta\zeta}\left(\frac{\partial \sigma_{32}}{\partial \theta} - \sigma_{22} + \sigma_{33}\right) + \eta\rho\omega^2 R v_3 = 0, \tag{2}$$

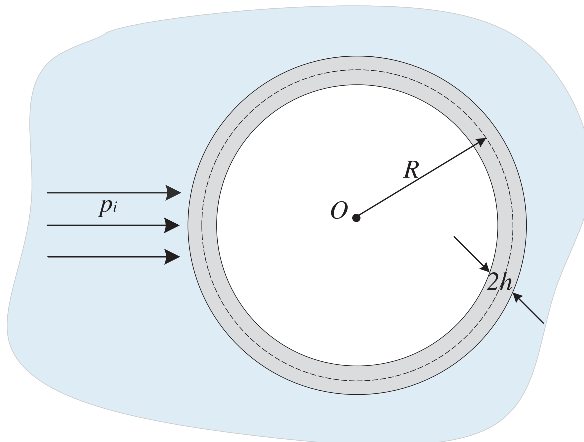

Figure 1. Thin elastic shell immersed in fluid.

where

$$\sigma_{22} = \frac{E}{(1-\nu^2)R}\frac{1}{1+\eta\zeta}\left(\frac{\partial v_2}{\partial\theta} + v_3\right) + \frac{\nu}{1-\nu}\sigma_{33} \tag{3}$$

$$\frac{E}{R}\frac{\partial v_3}{\partial\zeta} = \eta(1-\nu^2)\sigma_{33} - \eta\nu(1+\nu)\sigma_{22}, \tag{4}$$

and

$$\eta\sigma_{32} = \frac{E}{2(1+\nu)R}\left(\frac{\partial v_2}{\partial\zeta} - \frac{\eta}{1+\eta\zeta}\left(v_2 - \frac{\partial v_3}{\partial\theta}\right)\right), \tag{5}$$

here, σ_{ij} and v_i, $i,j = 2,3$ are the stress and displacement components, respectively, E is the Young modulus, ν is the Poisson ratio, and ρ is the mass density.

The fluid is governed by the 2D Helmholtz equation:

$$\Delta p + \frac{\omega^2}{c_0^2}p = 0, \tag{6}$$

where p is the acoustic pressure, c_0 is the sound speed, and Δ is the Laplace operator given by:

$$\Delta = \frac{1}{R^2}\left(\frac{1}{(1+\eta\zeta)^2}\frac{\partial^2}{\partial\theta^2} + \frac{1}{\eta(1+\eta\zeta)}\frac{\partial}{\partial\zeta} + \frac{1}{\eta^2}\frac{\partial^2}{\partial\zeta^2}\right).$$

The fluid pressure is the sum of a prescribed incident component p_i and a sought-for scattered one p_s, i.e.:

$$p = p_i + p_s. \tag{7}$$

In this paper, we concentrate on the analysis of the nth term in the trigonometric series for a plane wave (e.g., see [1]). It is:

$$p_i = p_0 J_n\left(\frac{\omega R(1+\eta\zeta)}{c_0}\right)\cos(n\theta) \quad \text{and} \quad p_s = p_0 S H_n^{(1)}\left(\frac{\omega R(1+\eta\zeta)}{c_0}\right)\cos(n\theta), \tag{8}$$

where $J_n(z)$ is the Bessel function and $H_n^{(1)}(z)$ is the Hankel function of the first kind (see Abramowitz and Stegun [18]), p_0 is a given constant, and S is an unknown coefficient to be determined from the solution of the problem. Here and below, for brevity, we do not supply the coefficients in the Fourier series with the suffix n.

The fluid–structure interaction along the interface $\zeta = 1$ is described through the boundary conditions:

$$\sigma_{32} = 0, \qquad \sigma_{33} = -p, \tag{9}$$

and

$$v_3 = \frac{1}{h\rho_0\omega^2}\frac{\partial p}{\partial \zeta}, \tag{10}$$

where ρ_0 denotes the fluid density. We also impose homogeneous boundary conditions at the inner face of the shell $\zeta = -1$ in the form:

$$\sigma_{3i} = 0, \qquad i = 2, 3. \tag{11}$$

In what follows, we focus on the first few shell resonances ($n \sim 1$) over the low-frequency range $\omega \sim c_2\eta^{3/2}/R$, with $c_2 = \sqrt{E/2(1+\nu)\rho}$ denoting the shear wave speed in the elastic material inspired by a recent eigenvalue analysis of a free shell [13]. The aim of this paper is to derive explicit asymptotic expansions for the resonance frequencies, amplitudes, and widths according to the framework of the resonance scattering theory (e.g., see [1,3]). Until now, such formulae were known only for higher frequencies (see Kaplunov et al. [2] and references therein).

3. Scaling

First, we define the dimensionless frequency in accordance with the assumption made in the previous section as:

$$\Omega = \frac{\omega R}{\eta^{3/2}c_2} \sim 1. \tag{12}$$

It is worth mentioning that for a free shell, i.e., one not in contact with a fluid, we have $\omega R/c_2 \sim \eta$ instead of equation (12) (e.g., see Ege et al. [13]). This is why, the analyzed phenomena may be referred to as a fluid-borne wave [6]. Recently, an asymptotic analysis of a fluid-borne wave on a flat layer was developed in Kaplunov et al. [14] on the basis of plane strain elasticity.

Then, we present the displacement and stress components, $u_i(\theta, \zeta)$ and $\sigma_{ij}(\theta, \zeta)$, and the pressure $p(\theta, \zeta) = p_i(\theta, \zeta) + p_s(\theta, \zeta)$ as:

$$v_2 = u_2 \sin n\theta, \qquad v_3 = u_3 \cos n\theta, \tag{13}$$

$$\sigma_{22} = s_{22}\cos n\theta, \quad \sigma_{32} = s_{32}\sin n\theta, \quad \sigma_{33} = s_{33}\cos n\theta, \tag{14}$$

and

$$p = P\cos n\theta, \tag{15}$$

where

$$P = P_i + P_s \tag{16}$$

and

$$p_i = P_i\cos n\theta, \quad p_s = P_s\cos n\theta, \tag{17}$$

with $u_i(\zeta)$, $s_{ij}(\zeta)$, $P_i(\zeta)$, $P_s(\zeta)$, and $P(\zeta)$ denoting the unknown Fourier coefficients.

Substituting expression (8) into the impenetrability condition (10), taking into account the second of the contact conditions (9) and using formulae (13)–(15), the remaining two contact conditions may be combined into a single one at $\zeta = 1$ as:

$$u_3 + \frac{\mathcal{H}}{\rho_0 c_0\omega}s_{33} = \frac{\mathcal{F}}{\rho_0 c_0 \omega}p_0, \tag{18}$$

where

$$\mathcal{H} = \frac{\left(H_n^{(1)}(z)\right)'}{H_n^{(1)}(z)}, \qquad \mathcal{F} = (\mathcal{J} - \mathcal{H})J_n, \tag{19}$$

with:

$$\mathcal{J} = \frac{J_n'(z)}{J_n(z)}. \tag{20}$$

In the formulae above, $z = \omega R(1+\eta)/c_0$, and a prime denotes a derivative with respect to the relevant argument.

Let us, now, introduce the dimensionless quantities similar to those for a free shell, e.g., see Ege et al. [13], as:

$$u_2 = Ru_2^*, \quad u_3 = Ru_3^* \tag{21}$$

and

$$s_{22} = E\eta s_{22}^*, \quad s_{32} = E\eta^2 s_{32}^*, \quad s_{33} = E\eta^2 s_{33}^*, \tag{22}$$

together with:

$$P(1) = E\eta^3 P^*, \quad P_i(1) = E\eta^3 P_i^*, \quad P_s(1) = E\eta^3 P_s^*, \tag{23}$$

and also:

$$p_0 = \rho_0 c_2^2 \eta^{3(2-n)/2} P_0^*. \tag{24}$$

In addition, we set:

$$\mathcal{H} = \eta^{-3/2}\mathcal{H}^*, \quad \mathcal{F} = \eta^{3(n-1)/2}\mathcal{F}^*, \tag{25}$$

where

$$\mathcal{H}^* = \frac{1}{\Omega}\left(1 - \eta + \eta^2 + \cdots\right)\Bigg(-nc_* - \frac{n-2}{4(n-1)c_*}\Omega^2(1+\eta)^2\eta^3 - \cdots + \frac{i\pi\,\Omega^{2n}}{2^{2n-1}c_*^{2n-1}((n-1))^2}(1+2n\eta)\eta^{3n} + \cdots\Bigg) \tag{26}$$

and

$$\mathcal{F}^* = (1 - \eta + \eta^2 + \cdots)\Bigg(\frac{2n\Omega^{n-1}(1+n\eta)}{c_*^{n-1}} - \frac{(n-2)\Omega^{n+1}(1+\eta)(1+n\eta)\eta^3}{4(n-1)c_*^{n+1}} + \cdots + \frac{i\pi\,\Omega^{3n-1}}{2^{2n-1}c_*^{3n-1}((n-1))^2}(1+3n\eta)\eta^{3n} + \ldots\Bigg), \tag{27}$$

where $c_* = c_0/c_2$, $\rho_* = \rho_0/\rho$, and all other quantities above marked by an asterisk are assumed to be of the same asymptotic order. In expansions (26) and (27), higher-order terms are not written out explicitly apart from an imaginary term in equation (27) corresponding to the radiation of the vibration energy into the fluid.

The boundary conditions given by equations (9) and (11), now, become:

$$s_{32}^* = 0, \quad \text{at} \quad \zeta = \pm 1, \quad s_{33}^* = 0 \quad \text{at} \quad \zeta = -1, \tag{28}$$

and

$$s_{33}^* = -\eta(P_i^* + P_s^*) \qquad \text{at} \quad \eta = 1. \tag{29}$$

The impenetrability condition (18) takes the form:

$$\Omega\eta u_3^* + \frac{2(1+\nu)}{c_*\rho_*}s_{33}^*\mathcal{H}^* = \frac{\eta}{c_*}\mathcal{F}^* P_0^*, \qquad \text{at} \quad \zeta = 1. \tag{30}$$

Inserting scalings (21) and (22) in the equations of motion (1) and (2) and the constitutive relations (3)–(5), we have:

$$\frac{\partial s_{32}^*}{\partial \zeta} - \frac{n}{1+\eta\zeta}s_{22}^* + \frac{2\eta}{1+\eta\zeta}s_{32}^* + \frac{\eta^2\Omega^2}{2(1+\nu)}u_2^* = 0, \tag{31}$$

$$\frac{\partial s_{33}^*}{\partial \zeta} + \frac{n\eta}{1+\eta\zeta}s_{32}^* - \frac{1}{1+\eta\zeta}s_{22}^* + \frac{\eta}{1+\eta\zeta}s_{33}^* + \frac{\eta^2\Omega^2}{2(1+\nu)}u_3^* = 0, \tag{32}$$

and

$$\eta s_{22}^* = \frac{1}{1-\nu^2}\left(\frac{n}{1+\eta\zeta}u_2^* + \frac{1}{1+\eta\zeta}u_3^*\right) + \frac{\nu}{1-\nu}\eta^2 s_{33}^*, \tag{33}$$

$$\frac{\partial u_3^*}{\partial \zeta} = (1-\nu^2)\eta^3 s_{33}^* - \nu(1+\nu)\eta^2 s_{22}^*, \tag{34}$$

$$\eta^3 s_{32}^* = \frac{1}{2(1+\nu)}\left(\frac{\partial u_2^*}{\partial \zeta} - \frac{n\eta}{1+\eta\zeta}u_3^* - \frac{\eta}{1+\eta\zeta}u_2^*\right). \tag{35}$$

In what follows, all quantities marked with an asterisk are expanded in asymptotic series as:

$$f^* = f^{(0)} + \eta f^{(1)} + \eta^2 f^{(2)} + \cdots \tag{36}$$

taking into account $(1+\eta\zeta)^{-1} = 1 - \eta\zeta + \eta^2\zeta^2 + \cdots$.

4. Asymptotic derivation

First, integrating equations (34) and (35) with respect to the thickness coordinate ζ and taking into consideration (33), we obtain, at leading order:

$$u_3^{(0)} = w^{(0)}, \quad u_2^{(0)} = v^{(0)}, \quad \text{and} \quad nv^{(0)} + w^{(0)} = 0, \tag{37}$$

where $w^{(0)}$ and $v^{(0)}$ are the unknown constants to be determined. The last formula in equation (37) demonstrates that the shell mid-surface appears to be inextensible at leading order. This is in line with an assumption underlying the semi-membrane theory for cylindrical shells (see [10,11]).

Next, integrating equations (31) and (32), we have:

$$s_{32}^{(0)} = -n\int_\zeta^1 s_{22}^{(0)}\,ds, \qquad s_{33}^{(0)} = -\int_\zeta^1 s_{22}^{(0)}\,ds. \tag{38}$$

Using the homogeneous boundary conditions (28) and (30), taking the form $s_{3i}^{(0)} = 0$, $i = 2, 3$, we conclude that:

$$\int_{-1}^1 s_{22}^{(0)}ds = 0. \tag{39}$$

It is clear that the frequency parameter does not appear in the relations above; therefore, we need to proceed with the next-order approximation. Then, we consider equations (34) and (35) at the first order inserting into the latter the derived expressions (33). Next, integrating along the thickness, we obtain:

$$u_3^{(1)} = w^{(1)}, \quad u_2^{(1)} = -\frac{1-n^2}{n}\zeta w^{(0)} + v^{(1)} \quad \text{and} \quad nv^{(1)} + w^{(1)} = 0, \tag{40}$$

where we also employ the relation (39).

Now, revisiting of equation (33), using relations $(40)_2$, $(40)_3$, and also equation (37), results in:

$$s_{22}^{(0)} = -\frac{1-n^2}{1-\nu^2}\zeta w^{(0)}. \tag{41}$$

As a result, expressions (38) may be rewritten as:

$$s_{32}^{(0)} = \frac{n(1-n^2)}{2(1-\nu^2)}\left(1-\zeta^2\right)w^{(0)}, \qquad s_{33}^{(0)} = \frac{1-n^2}{2(1-\nu^2)}\left(1-\zeta^2\right)w^{(0)}. \tag{42}$$

In a similar manner, we deduce from equation (31), integrating in the thickness variable and adopting equations (41) and (42), together with the appropriate boundary condition at $\zeta = 1$:

$$s_{32}^{(1)} = -n\int_\zeta^1 s_{22}^{(1)}\,ds + \frac{n(1-n^2)}{3(1-\nu^2)}\left(1-3\zeta+2\zeta^3\right)w^{(0)} \tag{43}$$

and

$$\int_{-1}^{1} s_{22}^{(1)}\, ds = \frac{2(1-n^2)}{3(1-\nu^2)} w^{(0)}. \tag{44}$$

Finally, integrating equation (32) in ζ, we arrive at:

$$s_{33}^{(1)} = -\int_{\zeta}^{1} s_{22}^{(1)}\, ds - \frac{(1-n^2)(1-\zeta)\left(3\zeta(1+\zeta) - n^2(2-\zeta-\zeta^2)\right)}{6(1-\nu^2)} w^{(0)}$$
$$+ \frac{\rho_*}{2(1+\nu)\mathcal{H}^{(0)}} \left(\mathcal{F}^{(0)} P_0^* - c_* \Omega w^{(0)}\right). \tag{45}$$

Here, we adapt the impenetrability condition (30) which, at first order, takes the form:

$$s_{33}^{(1)} = \frac{\rho_*}{2(1+\nu)\mathcal{H}^{(0)}} \left(\mathcal{F}^{(0)} P_0^* - c_* \Omega w^{(0)}\right) \quad \text{at} \quad \zeta = 1, \tag{46}$$

where

$$\mathcal{F}^{(0)} = \frac{2n\Omega^{n-1}}{c_*^{n-1}}, \qquad \mathcal{H}^{(0)} = -\frac{nc_*}{\Omega} + i\frac{\gamma}{\Omega} \tag{47}$$

and

$$\gamma = \frac{\pi\, \Omega^{2n}}{2^{2n-1} c_*^{2n-1} \left((n-1)!\right)^2} \eta^{3n}. \tag{48}$$

Equation (45), at $\zeta = -1$, reduces to:

$$\int_{-1}^{1} s_{22}^{(1)} d\zeta = \frac{2n^2(1-n^2)}{3(1-\nu^2)} w^{(0)} - \frac{\rho_*}{2(1+\nu)\mathcal{H}^{(0)}} \left(\mathcal{F}^{(0)} P_0^* - c_* \Omega w^{(0)}\right). \tag{49}$$

On comparing equations (44) and (49), we finally obtain the leading order vertical displacement given by:

$$w^{(0)} = \frac{C_0}{D_0}, \tag{50}$$

where the coefficients C_0 and D_0 are:

$$C_0 = \frac{\rho_* \mathcal{F}^{(0)} P_0^*}{2(1+\nu)\mathcal{H}^{(0)}}, \qquad D_0 = \frac{2(1-n^2)^2}{3(1-\nu^2)} + \frac{c_* \rho_*}{2(1+\nu)\mathcal{H}^{(0)}}\Omega. \tag{51}$$

Similarly, at the second order, we start by integrating equations (34) and (35) in ζ. Using equations (37), (40), and (41), we obtain:

$$u_3^{(2)} = \frac{\nu}{2(1-\nu)}(1-n^2)\zeta^2 w^{(0)} + w^{(2)}, \qquad u_2^{(2)} = -\frac{1-n^2}{n}\zeta w^{(1)} + v^{(2)}. \tag{52}$$

Next, integration of equation (33), taking into account equations (37), (40), $(42)_2$, and (52) leads to:

$$v^{(2)} + \frac{1}{n} w^{(2)} = -\frac{\nu(1-n^2)}{2(1-\nu)n} w^{(0)}. \tag{53}$$

Substituting the last formula back into equation (33), we get:

$$s_{22}^{(1)} = -\frac{1-n^2}{1-\nu^2}\zeta w^{(1)} + \frac{1-n^2}{1-\nu^2}\zeta^2 w^{(0)}. \tag{54}$$

This expression allows us to rewrite formulae (43) and (45) as:

$$s_{32}^{(1)} = \frac{n(1-n^2)}{2(1-\nu^2)}(1-\zeta^2)w^{(1)} - \frac{n(1-n^2)}{2(1-\nu^2)}(1-\zeta^2)\zeta w^{(0)} \tag{55}$$

and

$$s_{33}^{(1)} = \frac{(1-n^2)}{2(1-\nu^2)}(1-\zeta^2)w^{(1)} + w^{(0)} + \frac{\rho_*}{2(1+\nu)\mathcal{H}^{(0)}}\left(\mathcal{F}^{(0)}P_0^* - c_*\Omega w^{(0)}\right). \tag{56}$$

Then, we derive from equations (31) and (32), respectively:

$$\int_{-1}^{1} s_{22}^{(2)} ds = \frac{2(1-n^2)}{3(1-\nu^2)}w^{(1)} - \frac{1}{(1+\nu)n^2}\Omega^2 w^{(0)} \tag{57}$$

and

$$\begin{aligned}\int_{-1}^{1} s_{22}^{(2)} ds =& \frac{\rho_*}{(1+\nu)\mathcal{H}^{(0)}}\left(\mathcal{F}^{(0)}P_0^* - c_*\Omega w^{(0)}\right)\\ &+ \frac{\rho_*}{2(1+\nu)\mathcal{H}^{(0)}}\left(n\mathcal{F}^{(0)}P_0^* - c_*\Omega\left(w^{(0)} + w^{(1)}\right)\right)\\ &+ \frac{1}{1+\nu}\Omega^2 w^{(0)} + \frac{2n^2(1-n^2)}{3(1-\nu^2)}w^{(1)} - \frac{2(1-n^2)^2}{3(1-\nu^2)}w^{(0)}.\end{aligned} \tag{58}$$

Here, we apply the boundary conditions (28) together with equation (30) which, at the second order, becomes:

$$s_{33}^{(2)} = \frac{\rho_* P_0^* \mathcal{F}^{(1)}}{2(1+\nu)\mathcal{H}^{(0)}} - \frac{\mathcal{H}^{(1)}}{\mathcal{H}^{(0)}}s_{33}^{(1)} - \frac{c_*\rho_*\Omega}{2(1+\nu)\mathcal{H}^{(0)}}w^{(1)} \tag{59}$$

with:

$$\mathcal{F}^{(1)} = (n-1)\mathcal{F}^{(0)} \qquad \text{and} \qquad \mathcal{H}^{(1)} = \frac{nc_*}{\Omega} + \frac{i\gamma}{\Omega}. \tag{60}$$

As a result, equation (59), using equations (46) and (60), may be rewritten as:

$$s_{33}^{(2)} = \frac{\rho_*}{2(1+\nu)\mathcal{H}^{(0)}}\left(n\mathcal{F}^{(0)}P_0^* - c_*\Omega\left(w^{(0)} + w^{(1)}\right)\right), \quad \text{at} \quad \zeta = 1. \tag{61}$$

Finally, comparing equations (57) and (58), we obtain:

$$w^{(1)} = \frac{C_1}{D_1}, \tag{62}$$

where the coefficients C_1 and D_1 are given by:

$$C_1 = \frac{(n+2)\rho_*\mathcal{F}^{(0)}P_0^*}{2(1+\nu)}, \qquad D_1 = \frac{2(1-n^2)^2}{3(1-\nu^2)} + \frac{3c_*\rho_*}{2(1+\nu)\mathcal{H}^{(0)}}\Omega - \frac{(1+n^2)}{n^2(1+\nu)}\Omega^2. \tag{63}$$

5. Resonant scattering analysis

Let us define $W = w^{(0)} + \eta w^{(1)}$ starting from equations (50) and (62) in the previous two sections. Taking into account equations (51) and (63), we obtain:

$$W = \frac{2n\Omega^n p_0}{\rho_0 c_2^2 c_*^n}\frac{1+(n+2)\eta}{\Omega^2 - \Omega_0^2 - \eta\Omega_1^2 + i\Omega^2\gamma/(nc_*)}\eta^{3(n-2)/2}, \tag{64}$$

where

$$\Omega_0^2 = \frac{4n(1-n^2)^2}{3\rho_*(1-\nu)}, \qquad \Omega_1^2 = -\frac{8(1-n^2)^2(1+n\rho_*+n^2)}{3\rho_*^2(1-\nu)}, \tag{65}$$

with $\Omega^2 = \Omega_0^2 + \eta\Omega_1^2 + \ldots$ is the two-term expansion of the resonant frequency. It is worth noting that the dimensional counterpart of the leading order term $\omega_0^2 = c_2^2\eta^3\Omega_0^2/R^2$ does not depend on the shell density ρ, i.e., this is indeed a fluid-borne vibration, whereas both terms ω_m^2, $m = 0, 1$ do not depend on fluid compressibility manifesting through presence of fluid wave speed c_0.

In the vicinity of $\eta \ll |\Omega - \Omega_0| \ll 1$, equation (64), at leading order, takes the form:

$$W = A\frac{\Gamma}{\Omega - \Omega_0 + i\Gamma}, \tag{66}$$

where Γ is the resonance width of the modal resonance having the form:

$$\Gamma = \frac{\gamma_0\Omega_0}{2nc_*} \tag{67}$$

with $\gamma_0 = \gamma(\Omega_0)$ and A is its amplitude given by:

$$A = \frac{2^{2n}c_*^n(n!)^2}{\pi\rho_0 c_2^2\,\Omega_0^{n+2}}\eta^{-3(n+2)/2} \tag{68}$$

Let us also determine the unknown coefficient S at leading order. Using boundary condition $(9)_2$ together with relations $(14)_3$–(17) at $\zeta = 1$, we have:

$$s_{33} = -(P_i + P_s). \tag{69}$$

Employing equations $(22)_3$–(25), (29), (46), and using the fact that $s_{33}^{(0)}$ is identically zero at $\zeta = 1$, we obtain:

$$s_{33}^{(1)} = -\frac{p_0}{E\eta^3}\left(J_n + SH_n^{(1)}\right). \tag{70}$$

Substituting equations (19), (20), (24), (25), and (46) into the last formula, we arrive at:

$$S = S_b + S_r, \tag{71}$$

with:

$$S_b = -\frac{J_n'}{\left(H_n^{(1)}\right)'} \tag{72}$$

and

$$S_r = -\frac{2n\left(1 + i\gamma/nc_*\right)}{c_*^{n-1}\left(\Omega_0^2 - \Omega^2\left(1 + i\gamma/nc_*\right)\right)\left(H_n^{(1)}\right)'}\,\Omega^{n+1}\eta^{3(n-1)/2}, \tag{73}$$

where we make use of equation (50) in the last relation. Here, S_b is the background component corresponding to the pressure scattered by a rigid cylinder, while S_r is the resonant component specific to an elastic hollow cylinder. This derivation proves that over the studied low-frequency domain, a rigid cylinder is the optimal background. Before, the choice of background was usually based on intuitive assumptions (e.g., see Veksler [1,19]).

We may easily derive the leading order terms in equations (72) and (73) (see Abramowitz and Stegun [18]), which are evaluated as:

$$S_b = \frac{i\pi\Omega^{2n}}{2^{2n}c_*^{2n}n!(n-1)!}\eta^{3n} \tag{74}$$

and

$$S_r = \frac{i\pi\left(1 + i\,\gamma/(nc_*)\right)\Omega^{2n+2}}{2^{n-1}c_*^{2n}(n-1)!}\left(\frac{1}{\Omega_0^2 - \Omega^2\left(1 + i\gamma/(nc_*)\right)}\right)\eta^{3n}. \tag{75}$$

Again, in the vicinity $\eta \ll |\Omega - \Omega_0| \ll 1$, the latter takes the form:

$$S_r = A_r\frac{\Gamma}{\Omega - \Omega_0 + i\Gamma}, \tag{76}$$

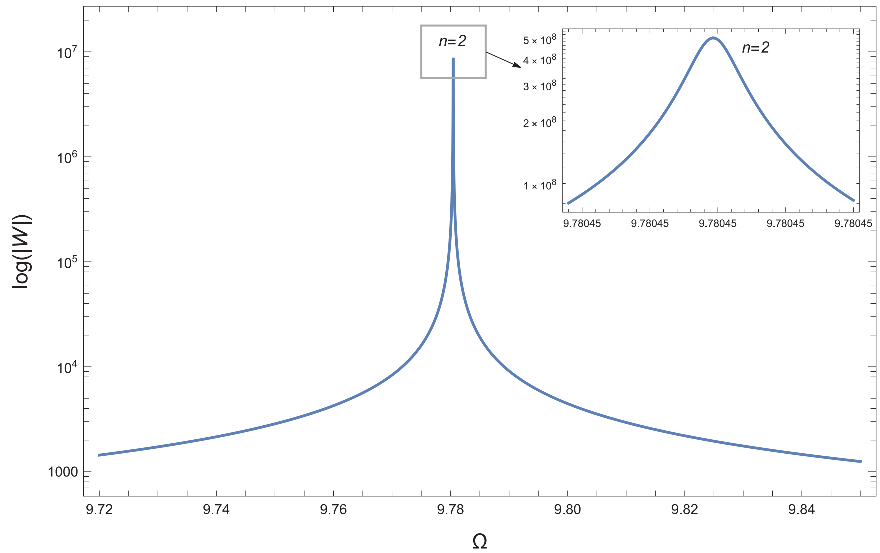

Figure 2. Displacement (66) for $n = 2$ with the Poisson ratio $\nu = 0.3$ and $\eta = 0.01$.

Table 1. Numerical values of resonance scattering characteristics for $n = 2, 3, 4$, $\nu = 0.3$, and $\eta = 0.01$.

n	Ω_0	γ_0	Γ	A	$\lvert A_r \rvert$
2	9.78	3.36×10^{-8}	1.73×10^{-7}	5.006×10^{8}	8
3	31.94	1.08×10^{-9}	1.22×10^{-8}	2.351×10^{9}	48
4	69.15	6.61×10^{-11}	1.20×10^{-9}	2.169×10^{10}	384

with the amplitude A_r given by:

$$A_r = -i\, 2^n n!. \tag{77}$$

Figure 2 demonstrates the resonant behavior of an aluminum cylinder submerged in water for the displacement W at $n = 2$ (see equation (66) with equations (67) and (68)). The problem parameters are $\rho = 2790$ kg/m^3, $c_2 = 3100$ m/s, $\rho_0 = 1000$ kg/m^3, $c_0 = 1470$ m/s, $\nu = 0.3$, and $\eta = 0.01$. The plot is presented in the log-scale with the amplitude A in equation (68) normalized by $A\rho_0 c_2^2/p_0$. The asymptotic formula (68) reveals that the displacement amplitude grows at n at the fixed small parameter η (see also numerical data for $n = 2, 3, 4$ in Table 1).

In Figure 3, we present a similar curve for the resonance component of the scattered pressure starting from equation (76) with equations (77) and (67). In this case, the estimation (77) indicates the growth of the amplitude $|A_r|$ in n (see also Table 1). At the same time, in contrast to equation (68), it grows not as the power of the small parameter η.

The formulae (67) and (48) as well as the relevant column in Table 1 show that the widths of the observed resonances are $O(\eta^{3n})$ diminishing as n grows. This is why, we supply Figures 2 and 3 by "microscopic" plots providing a better resolution. However, these plots only illustrate the form of the resonance curves but not their exact location. Indeed, for them $|\Omega - \Omega_0| \sim \eta^{3n}$, i.e., their maxima are outside the chosen vicinity of the resonance frequency, where $|\Omega - \Omega_0| \gg \eta$. The last strong inequality ensures the validity of the adapted leading order approximation in the expansion (64).

6. Conclusion

The problem of acoustic scattering by a circular cylindrical shell at normal incidence is revisited assuming that the shell motion is governed by the "exact" plane strain equation in linear elasticity. A high-order asymptotic scheme, oriented to low-frequency motion of a thin fluid-loaded shell over the range (12), is developed. The resonances of a fluid-borne wave are investigated according to the Resonance Scattering Theory [1].

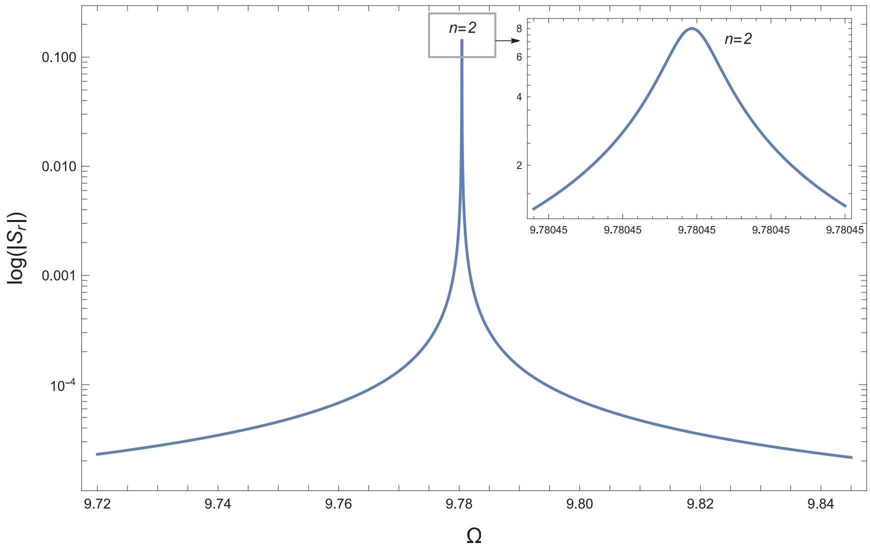

Figure 3. Resonance component (76) for $n = 2$ with $\nu = 0.3$ and $\eta = 0.01$.

It is demonstrated that a background component has to be taken as the pressure scattered by a rigid cylinder (see equation (72)). Explicit formulae are obtained for resonant frequencies, amplitudes, and widths (see equations (65), (67), (68), and (77)).

These formulae allow evaluation of remarkably narrow resonant peaks (see Figures 2 and 3 along with the Table 1). Numerical analysis of such peaks without a preliminary asymptotic insight is highly problematic. Indeed, their width (equations (66) and (76)) is significantly less than the error of the asymptotic expansion (65) (see Veksler [19]). Certainly, the derivation of more accurate expansions would be very time-consuming.

Apparently, a more realistic prediction for the analyzed peaks might be established by taking into account structural damping. The latter also seems to be useful for numerical experiments, providing an initial approximation even for a purely elastic scenario. In this case, it would be natural to begin with inserting a damping coefficient into the exact elastic solution.

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References

[1] Veksler, ND. *Resonance acoustic spectroscopy*, vol. 11. Berlin: Springer Science & Business Media, 2012.
[2] Kaplunov, JD, Kossovitch, LY, and Nolde, EV. *Dynamics of thin walled elastic bodies*. Cambridge, MA: Academic Press, 1998.
[3] Gaunaurd, G, and Werby, M. Acoustic resonance scattering by submerged elastic shells. *Appl Mech Rev* 1990; 43(8): 171–208.
[4] Zarastvand, M, Ghassabi, M, and Talebitooti, R. Acoustic insulation characteristics of shell structures: a review. *Arch Comput Method Eng* 2021; 28: 505–523.
[5] Uginčius, P, and Überall, H. Creeping-wave analysis of acoustic scattering by elastic cylindrical shells. *J Acoust Soc Am* 1968; 43(5): 1025–1035.
[6] Talmant, M, Überall, H, Miller, RD, et al. Lamb waves and fluid-borne waves on water-loaded, air-filled thin spherical shells. *J Acoust Soc Am* 1989; 86(1): 278–289.
[7] Veksler, N. The analysis of peripheral waves in the problem of plane acoustic pressure wave scattering by a circular cylindrical shell. *Acta Acust United Acust* 1989; 69(2): 63–72.
[8] Belov, A, Kaplunov, J, and Nolde, E. A refined asymptotic model of fluid-structure interaction in scattering by elastic shells. *Flow Turbul Combust* 1998; 61(1–4): 255–267.
[9] Goldenveizer, A. *Theory of thin elastic shells* (Russian book). Moscow: Izdatel'stvo Nauka, 1976.
[10] Calladine, CR. *Theory of shell structures*. Cambridge: Cambridge University Press, 1983.

[11] Goldenveizer, A. *Theory of elastic thin shells: solid and structural mechanics*, vol. 2. Amsterdam: Elsevier, 2014.
[12] Ege, N, Erbaş, B, Kaplunov, J, et al. Asymptotic corrections to the low-frequency theory for a cylindrical elastic shell. *Z Angew Math Phys* 2023; 74(2): 43.
[13] Ege, N, Erbaş, B, and Kaplunov, J. Asymptotic derivation of refined dynamic equations for a thin elastic annulus. *Math Mech Solids* 2021; 26(1): 118–132.
[14] Kaplunov, J, Prikazchikova, L, and Shamsi, S. A hierarchy of asymptotic models for a fluid-loaded elastic layer. *Math Mech Solids*. Epub ahead of print 4 November 2023. DOI: 10.1177/10812865231201573.
[15] Krynkin, A, Umnova, O, Taherzadeh, S, et al. Analytical approximations for low frequency band gaps in periodic arrays of elastic shells. *J Acoust Soc Am* 2013; 133(2): 781–791.
[16] Krynkin, A, Umnova, O, Vicente Sanchez-Perez, J, et al. Acoustic insertion loss due to two-dimensional periodic arrays of circular cylinders parallel to a nearby surface. *J Acoust Soc Am* 2011; 130(6): 3736–3745.
[17] Xu, S, Qiu, C, and Liu, Z. Acoustic transmission through asymmetric grating structures made of cylinders. *J Appl Phys* 2012; 111(9): 094505.
[18] Abramowitz, M, and Stegun, IA. *Handbook of mathematical tables*. Mineola, NY: Dover Publications, 1968.
[19] Veksler, N. Intermediate background in problems of sound waves scattering by elastic shells. *Acta Acust United Acust* 1992; 76(1): 1–9.