



On ‘rotating charged AdS solutions in quadratic $f(T)$ gravity’: new rotating solutions

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Abstract We show that there are two or more procedures to generalize the known four-dimensional transformation, aiming to generate cylindrically rotating charged exact solutions, to higher dimensional spacetimes. In the one procedure, presented in Eur. Phys. J. C (2019) 79:668, one uses a non-trivial, non-diagonal, Minkowskian metric $\tilde{\eta}_{ij}$ to derive complicated rotating solutions. In the other procedure, discussed in this work, one selects a diagonal Minkowskian metric η_{ij} to derive much simpler and appealing rotating solutions. We also show that if $(g_{\mu\nu}, \eta_{ij})$ is a rotating solution then $(\tilde{g}_{\mu\nu}, \tilde{\eta}_{ij})$ is a rotating solution too with similar geometrical properties, provided $\tilde{\eta}_{ij}$ and η_{ij} are related by a symmetric matrix R : $\tilde{\eta}_{ij} = \eta_{ik} R_{kj}$.

1 Preliminaries

In this work we will use the notation of [1] with a slight difference. Instead of taking $f(T) = T + \alpha T^2$ with $\alpha < 0$ we will take $f(T) = T - \alpha T^2$ with $\alpha > 0$.

Another different choice, which will be made clearer later, is the signature of the N -dimensional Minkowski spacetime: $(+, -, -, \dots)$. Most of the other notations will be almost similar to that of [1].

As a first comment we state that there are some sign mistakes in the definition of $K_{\alpha\mu\nu}$ of [1]. We use the following definitions:¹

¹ $S^{\alpha\mu\nu}$ may be given in a more compact form as:

$$S^{\alpha\mu\nu} = \frac{1}{4}(T^{\nu\mu\alpha} + T^{\alpha\mu\alpha} - T^{\mu\nu\alpha}) - \frac{1}{2}g^{\alpha\nu}T^{\sigma\mu}{}_{\sigma} + \frac{1}{2}g^{\alpha\mu}T^{\sigma\nu}{}_{\sigma}.$$

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$$\begin{aligned} T^{\alpha}{}_{\mu\nu} &= e_b{}^{\alpha}(\partial_{\mu}e^b{}_{\nu} - \partial_{\nu}e^b{}_{\mu}), \\ K_{\alpha\mu\nu} &= \frac{1}{2}(T_{\mu\alpha\nu} + T_{\nu\alpha\mu} - T_{\alpha\mu\nu}), \\ S^{\alpha\mu\nu} &= \frac{1}{2}(K^{\mu\nu\alpha} - g^{\alpha\nu}T^{\sigma\mu}{}_{\sigma} + g^{\alpha\mu}T^{\sigma\nu}{}_{\sigma}), \\ T &= T_{\alpha\mu\nu}S^{\alpha\mu\nu}. \end{aligned} \quad (1)$$

It is obvious from these definitions that the global sign of T would depend on the signature of the metric. For a static metric with signature $(+, -, -, -, \dots)$

$$ds^2 = A(r)dt^2 - \frac{1}{B(r)}dr^2 - r^2\left(\sum_{i=1}^n d\phi_i^2 + \sum_{i=1}^{N-n-2} \frac{dz_i^2}{l^2}\right), \quad (2)$$

where n is the number of angular coordinates, N is the dimension of spacetime and l is related to the cosmological constant by

$$\Lambda = -\frac{(N-1)(N-2)}{2l^2} < 0. \quad (3)$$

We obtain

$$T = +\frac{(N-2)A'B}{rA} + \frac{(N-2)(N-3)B}{r^2}. \quad (4)$$

Had we reversed the signature of the metric we would obtain the same expression with the two ‘+’ signs changed to ‘-’ signs. A second comment is also in order: The expression of T given in [1] has an extra factor 2 in the term including A' .

A final comment: The last term in Eq. (14) of [1] should have the opposite global sign. Using our metric-signature choice, Eq. (14) of [1] takes the form

$$\begin{aligned}
 ds^2 = & A(r) \left(\Xi dt - \sum_{i=1}^n \omega_i d\phi_i \right)^2 - \frac{dr^2}{B(r)} - r^2 \sum_{i=1}^{N-n-2} \frac{dz_i^2}{l^2} \\
 & - \frac{r^2}{l^4} \sum_{i=1}^n (\omega_i dt - \Xi l^2 d\phi_i)^2 - \frac{r^2}{l^2} \sum_{i < j}^n (\omega_i d\phi_j - \omega_j d\phi_i)^2,
 \end{aligned} \tag{5}$$

where $(\omega_1, \omega_2, \dots, \omega_n)$ are the rotation n parameters, $(\phi_1, \phi_2, \dots, \phi_n)$ are the n angular coordinates and $\Xi = \sqrt{1 + \sum_{i=1}^n (\omega_i^2/l^2)}$. Note that the last term, $-(r^2/l^2) \sum_{i < j}^n (\omega_i d\phi_j - \omega_j d\phi_i)^2$, vanishes identically if the space-time has only one angular coordinate.

The field equations of Maxwell- $f(T)$ gravity are given in Eq. (3) of [1], which we rewrite here for convenience

$$\begin{aligned}
 I_\mu{}^\nu \equiv & S_\mu{}^{\rho\nu} \partial_\rho T f_{TT} + [e^{-1} e^a{}_\mu \partial_\rho (e e_a{}^\alpha S_\alpha{}^{\rho\nu}) - T^\alpha{}_{\lambda\mu} S_\alpha{}^{\nu\lambda}] f_T \\
 & - \frac{\delta_\mu^\nu}{4} \left(f + \frac{(N-1)(N-2)}{l^2} \right) + \frac{\kappa}{2} T_{(em)\mu}{}^\nu = 0, \\
 \partial_\nu (\sqrt{|g|} F^{\mu\nu}) = & 0,
 \end{aligned} \tag{6}$$

where $e \equiv \sqrt{|g|}$ and $T_{(em)\mu}{}^\nu = F_{\mu\alpha} F^{\nu\alpha} - \frac{1}{4} \delta_\mu^\nu F_{\alpha\beta} F^{\alpha\beta}$, with $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$, is the energy-momentum tensor of the electromagnetic field. Here the ratio $(N-1)(N-2)/l^2$ is proportional to the cosmological constant Λ (3). It is obvious from the shape of Eq. (6) that we are dealing with a spin-zero (pure tetrad) $f(T)$ gravity. The general field equations including spin connection terms are provided in [2]

A particular charged static solution to the field equations (6) with $f(T) = T - \alpha T^2$ and $\alpha = -1/(24\Lambda) > 0$ has been determined [3] and is given by Eqs. (8) of [1]

$$\begin{aligned}
 A(r) = & \frac{r^2}{6(N-1)(N-2)\alpha} - \frac{m}{r^{N-3}} + \frac{3(N-3)q^2}{(N-2)r^{2(N-3)}} \\
 & + \frac{2\sqrt{6\alpha}(N-3)^3 q^3}{(2N-5)(N-2)r^{3N-8}}, \\
 B(r) = & A(r) \left[1 + \frac{\sqrt{6\alpha}(N-3)q}{r^{N-2}} \right]^{-2}, \\
 \Phi(r) = & \frac{q}{r^{N-3}} + \frac{\sqrt{6\alpha}(N-3)^2 q^2}{(2N-5)r^{2N-5}},
 \end{aligned} \tag{7}$$

where we have replaced Λ_{eff} by $1/[6(N-1)(N-2)\alpha]$. Note that since $\alpha > 0$ we have $\Lambda_{\text{eff}} > 0$.

2 Generating cylindrically rotating charged exact solutions

Consider the following substitution where a denotes a rotation parameter

$$dt \rightarrow \sqrt{1 + \frac{a^2}{l^2}} dt - a d\phi, \quad d\phi \rightarrow \sqrt{1 + \frac{a^2}{l^2}} d\phi - \frac{a}{l^2} dt. \tag{8}$$

There is no claim whatsoever in Refs. [4,5] that the substitution (8) is a shortcut or a trick for generating rotating solutions from static ones, however, some authors have applied the substitution (8) as a procedure to generate their supposed-to-be rotating solutions. In this work we present a general comment on the transformation (8) and its generalization to higher dimensions.

Our starting point is the expression of the tetrad $e^i{}_\mu$ in terms of the static metric $(A(r), B(r))$, the n rotation parameters denoted by $(\omega_1, \omega_2, \dots, \omega_n)$ instead of (a_1, a_2, \dots, a_n) , and the constant $\Xi = \sqrt{1 + \sum_{i=1}^n (\omega_i^2/l^2)}$. The tetrad expression $e^i{}_\mu$ is given in Eq. (12) of [1]. However, in order to evaluate $e_i{}^\mu$ from $e^i{}_\mu$, using the expression $e_i{}^\mu = \eta_{ij} g^{\mu\nu} e^j{}_\nu$, we need an expression for the Minkowskian metric η_{ij} . The authors of Ref. [1] *did not provide any expression for η_{ij}* they used in their work. An anonymous referee claimed that it is the non-diagonal form of η_{ij} , as given in Eq. (44) of Ref. [8] and Eq. (41) of Ref. [9], that has been used in [1] and that it is the only valid form of η_{ij} to be used. In this work we will use two different expressions for η_{ij} and we shall show that *the statement of the referee does not hold true* by constructing a new cylindrically rotating charged solution using a *diagonal* expression for η_{ij} .

For $N = 4$ we have checked that the proposed rotating solution in [1] satisfies the field equations (6) with $\kappa = -2$ taking a diagonal Minkowskian metric $\eta_{ij} = \text{diag}(1, -1, -1, -1)$.

From now on we restrict ourselves to $N = 5$ and consider the cases 1) $n = 1$ and 2) $n = 2$.

2.1 Case (1) $N = 5, n = 1$

In this case the coordinates are denoted by (t, r, ϕ, z_1, z_2) . The tetrad expression (12) of [1] reduces to

$$e^i{}_\mu = \begin{pmatrix} \Xi\sqrt{A(r)} & 0 & -\omega\sqrt{A(r)} & 0 & 0 \\ 0 & \frac{1}{\sqrt{B(r)}} & 0 & 0 & 0 \\ \frac{\omega r}{l^2} & 0 & -\Xi r & 0 & 0 \\ 0 & 0 & 0 & \frac{r}{l} & 0 \\ 0 & 0 & 0 & 0 & \frac{r}{l} \end{pmatrix}. \tag{9}$$

This is not a proper tetrad as the associated spin connection does not vanish [2,6]. To evaluate the associated spin connection we refer to [2,6]. Using the terminology of these references, the reference tetrad $e^i_{(r)\mu}$ is, in this case, given by (9) upon setting $m = q = 0$ (absence of gravity and matter) and $N = 5$. We find that the nonvanishing components of the spin connection $\omega^a_{b\mu}$ are [the Latin indexes (a, b) in $\omega^a_{b\mu}$ run from $1 \rightarrow 5$]: $\omega^1_{2t} = \omega^2_{1t} = \Xi r / (72\alpha)$, $\omega^1_{2\phi} = \omega^2_{1\phi} = -\omega r / (72\alpha)$, $\omega^2_{3t} = \omega^3_{2t} = -\omega r / (6\sqrt{2\alpha}l^2)$, $\omega^2_{3\phi} = \omega^3_{2\phi} = -\Xi r / (6\sqrt{2\alpha}l^2)$, $\omega^2_{4z_1} = \omega^4_{2z_1} = \omega^2_{5z_2} = \omega^5_{2z_2} = -r / (6\sqrt{2\alpha}l)$. This fact results in violation of local Lorentz invariance.

Taking a diagonal Minkowskian metric $\eta_{ij} = \text{diag}(1, -1, -1, -1, -1)$, the corresponding metric $g_{\mu\nu} = \eta_{ij}e^i_{\mu}e^j_{\nu}$ reads

$$ds^2 = A(r)(\Xi dt - \omega d\phi)^2 - \frac{dr^2}{B(r)} - \frac{r^2}{l^4}(\omega dt - \Xi l^2 d\phi)^2 - \frac{r^2 dz_1^2}{l^2} - \frac{r^2 dz_2^2}{l^2}, \tag{10}$$

which is the same as the metric suggest in Eq. (14) of [1]; in this case ($N = 5, n = 1$) the last term in Eq. (14) of [1] vanishes identically.

Now, we evaluate T upon substituting (9) and (10) into (1) and the resulting expression is identical to (4) taking $N = 5$.

On substituting (9), (10) and (4) into the field equations (6) and using the static solution (7) we noticed that all the field equations are satisfied.

2.2 Case (2) $N = 5, n = 2$

In this case the coordinates are denoted by $(t, r, \phi_1, \phi_2, z)$. The tetrad expression (12) of [1] reduces to

$$(e^i_{\mu}) = \begin{pmatrix} \Xi\sqrt{A(r)} & 0 & -\omega_1\sqrt{A(r)} & -\omega_2\sqrt{A(r)} & 0 \\ 0 & \frac{1}{\sqrt{B(r)}} & 0 & 0 & 0 \\ \frac{\omega_1 r}{l^2} & 0 & -\Xi r & 0 & 0 \\ \frac{\omega_2 r}{l^2} & 0 & 0 & -\Xi r & 0 \\ 0 & 0 & 0 & 0 & \frac{r}{l} \end{pmatrix}. \tag{11}$$

In order to proof that Eq. (14) of Ref. [1], which is Eq. (5) of this work (including the global sign correction we made), is a rotating solution one needs an expression for the Minkowskian matrix η_{ij} by which one can evaluate e_i^{μ} from e^i_{μ} (11), then evaluate all the tensors needed in the field equations (6). We divide this case into two sub-cases a) η_{ij} diagonal and b) η_{ij} non-diagonal.

2.2.1 Case (a) η_{ij} diagonal

If $\eta_{ij} = \text{diag}(1, -1, -1, -1, -1)$, the corresponding metric $g_{\mu\nu} = \eta_{ij}e^i_{\mu}e^j_{\nu}$ takes the form

$$ds^2 = A(r)(\Xi dt - \omega_1 d\phi_1 - \omega_2 d\phi_2)^2 - \frac{dr^2}{B(r)} - \frac{r^2 dz^2}{l^2} - \frac{r^2}{l^4} \sum_{i=1}^2 (\omega_i dt - \Xi l^2 d\phi_i)^2. \tag{12}$$

This metric has been directly derived from the vielbein (11) and $\eta_{ij} = \text{diag}(1, -1, -1, -1, -1)$. It is different from the rotating metric suggested in Eq. (14) of [1], which is Eq. (5) of this work. The difference resides in the last term in Eq. (5) which, in this case ($N = 5, n = 2$), reduces to $-(r^2/l^2)(\omega_1 d\phi_2 - \omega_2 d\phi_1)^2$.

Knowing the metric we evaluate e_i^{μ} by $e_i^{\mu} = \eta_{ij}g^{\mu\nu}e^j_{\nu}$. Next, we evaluate T upon substituting (11) and (12) into (1) and the resulting expression is identical to (4) taking $N = 5$.

Now, on substituting (11), (12) and (4) into the field equations (6) and using the static solution (7) we noticed that all the field equations are satisfied.

We have thus obtained a new rotating solution given by (12), which we rewrite for convenience

$$ds^2 = A(r) \left(\Xi dt - \sum_{i=1}^2 \omega_i d\phi_i \right)^2 - \frac{dr^2}{B(r)} - \frac{r^2 dz^2}{l^2} - \frac{r^2}{l^4} \sum_{i=1}^2 (\omega_i dt - \Xi l^2 d\phi_i)^2. \tag{13}$$

This is a solution to the field equations (6) with e^i_{μ} given by (11), $\eta_{ij} = \text{diag}(1, -1, -1, -1, -1)$, $\Xi = \sqrt{1 + \sum_{i=1}^2 (\omega_i^2 / l^2)}$, $A_{\mu} dx^{\mu} = \Phi(r)(\Xi dt - \sum_{i=1}^2 \omega_i d\phi_i)$, and the r -functions (A, B, Φ) are given in (7).

2.2.2 Case (b) η_{ij} non-diagonal

The authors of Ref. [1] did not provide an expression for the Minkowskian metric η_{ij} they used in their work. In our first version of this work we assumed $\eta_{ij} = \text{diag}(1, -1, -1, -1, -1)$ and we reached the conclusion that the metric (5) is not a solution to the field equations (6). However, an anonymous referee claimed that a correct expression for η_{ij} would be the matrix (44) of Ref. [8], which is also the matrix (41) of Ref. [9]. The rightmost column and the bottom line of that matrix have a common element, which is -1 , and the rest of the elements of the rightmost column and the bottom line are 0. In the case of five-dimensional spacetime with 2 angular coordinates ($N = 5, n = 2$), matrix (44) of Ref. [8], or matrix (41)

of Ref. [9], takes the following form using the notation and signature of this work (The authors of Refs. [8,9] claim that the metric [14], which is matrix (44) of Ref. [8] and matrix (41) of Ref. [9], is the ‘Minkowskian metric in cylindrical coordinates’. **This is very confusing**, for the Minkowskian metric in cylindrical coordinates depends on the radial coordinate r while the metric [14] is constant and does not depend on r)

$$\eta_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 - \frac{\omega_2^2}{l^2 \Xi^2} & \frac{\omega_1 \omega_2}{l^2 \Xi^2} & 0 \\ 0 & 0 & \frac{\omega_1 \omega_2}{l^2 \Xi^2} & -1 - \frac{\omega_1^2}{l^2 \Xi^2} & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}. \tag{14}$$

With this η_{ij} matrix and the expression of $e^i{}_\mu$ given in (11), the formula $g_{\mu\nu} = \eta_{ij} e^i{}_\mu e^j{}_\nu$ yields the metric (5). It is straightforward to show that the metric (5), which we rewrite for convenience

$$ds^2 = A(r) \left(\Xi dt - \sum_{i=1}^2 \omega_i d\phi_i \right)^2 - \frac{dr^2}{B(r)} - \frac{r^2 dz^2}{l^2} - \frac{r^2}{l^4} \sum_{i=1}^2 (\omega_i dt - \Xi l^2 d\phi_i)^2 - \frac{r^2}{l^2} (\omega_1 d\phi_2 - \omega_2 d\phi_1)^2, \tag{15}$$

is a solution to the field equations (6) with $e^i{}_\mu$ given by (11), η_{ij} given by (14), $\Xi = \sqrt{1 + \sum_{i=1}^2 (\omega_i^2/l^2)}$, $A_\mu dx^\mu = \Phi(r) (\Xi dt - \sum_{i=1}^2 \omega_i d\phi_i)$, and the r -functions (A , B , Φ) are given in (7).

It is also straightforward to show that T , upon substituting (11) and (15) into (1), has the same expression as in (4) taking $N = 5$.

In concluding, there are two cylindrically rotating solutions to the field equations (6). The first solution, derived in this work (13), is much simpler and is used with a diagonal Minkowskian metric $\eta_{ij} = \text{diag}(1, -1, -1, -1, -1)$. The second solution (15), derived in Ref. [1] (with the global sign correction of its last term made in this work), includes *extra terms*, $-(r^2/l^2) \sum_{i < j}^n (\omega_i d\phi_j - \omega_j d\phi_i)^2$, the number of which depends on the number n of angular coordinates and is used with a non-diagonal Minkowskian metric η_{ij} (14).

It is not clear why the authors of Refs. [1,8,9] used a non-trivial, non-diagonal, Minkowskian metric (14) that they claim to be the ‘Minkowskian metric in cylindrical coordinates’. This has nothing to do with cylindrical coordinates! [see (The authors of Refs. [8,9] claim that the metric [14], which is matrix (44) of Ref. [8] and matrix (41) of Ref. [9], is the ‘Minkowskian metric in cylindrical coordinates’. **This is very confusing**, for the Minkowskian met-

ric in cylindrical coordinates depends on the radial coordinate r while the metric [14] is constant and does not depend on r) for details]. Moreover, such a non-diagonal Minkowskian metric has led to a more complicated rotating solution (15). As a consequence, the rotating solutions derived in [8,9] have the same complicated structure as the one derived in [1] and they can be simplified on removing the extra terms $\mp(r^2/l^2) \sum_{i < j}^n (\omega_i d\phi_j - \omega_j d\phi_i)^2$ provided they are used with a diagonal Minkowskian metric $\eta_{ij} = \pm \text{diag}(1, -1, -1, -1, \dots, -1)$.

A point to emphasize is that when evaluating the metric from the formula $g_{\mu\nu} = \eta_{ij} e^i{}_\mu e^j{}_\nu$ one has to use $\eta_{ij} = \pm \text{diag}(1, -1, -1, -1, \dots, -1)$ and not a non-diagonal expression. The tetrad defined in (11) forms a *trivial pseudo-Cartesian system with metric* $\eta_{ij} = \text{diag}(1, -1, -1, -1, \dots, -1)$. Another anonymous referee has supported our claim.

3 Non-diagonal solutions versus diagonal solutions

From now on, a non-diagonal Minkowskian metric will be denoted by $\bar{\eta}_{ij}$. Let $\bar{\eta}_{ij}$ and η_{ij} be a non-diagonal and a diagonal Minkowskian metrics of dimension N , respectively. These two metrics may be related by a *symmetric* matrix R ($R_{ij} = R_{ji}$) such that $\bar{\eta}_{ij} = \eta_{ik} R_{kj}$. For instance, $\bar{\eta}_{ij}$ given by (14) and $\eta_{ij} = \text{diag}(1, -1, -1, -1, -1)$ are related by $R_{ij} = \eta_{ik} \bar{\eta}_{kj}$:

$$R_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 + \frac{\omega_2^2}{l^2 \Xi^2} & -\frac{\omega_1 \omega_2}{l^2 \Xi^2} & 0 \\ 0 & 0 & -\frac{\omega_1 \omega_2}{l^2 \Xi^2} & 1 + \frac{\omega_1^2}{l^2 \Xi^2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{16}$$

Let $\bar{g}_{\mu\nu}$ and $g_{\mu\nu}$ be the corresponding spacetime metrics, respectively.

The purpose of this section is to show that if $(g_{\mu\nu}, \eta_{ij})$ is a rotating solution then $(\bar{g}_{\mu\nu}, \bar{\eta}_{ij})$ is a rotating solution too with similar geometrical properties. Using $\bar{g}_{\mu\nu} = \bar{\eta}_{ij} e^i{}_\mu e^j{}_\nu$ and the fact that $\bar{g}_{\mu\sigma} \bar{g}^{\sigma\nu} = \delta_\mu^\nu$ we obtain

$$\bar{g}^{\mu\nu} = \eta^{ik} R^{kj} e_i{}^\mu e_j{}^\nu, \tag{17}$$

where η^{ik} and R^{kj} are the inverse matrices of η_{ik} and R_{kj} , respectively. Next, we evaluate $\bar{e}_i{}^\mu = \bar{\eta}_{ij} \bar{g}^{\mu\nu} e^j{}_\nu$. Using the expression (17) of $\bar{g}^{\mu\nu}$ and the fact that R_{ij} is symmetric, we obtain

$$\bar{e}_i{}^\mu = e_i{}^\mu, \tag{18}$$

which along with the relation $\bar{e}^i{}_\mu = e^i{}_\mu$ (true by definition since we are using the same tetrad but different Minkoskian metrics) imply that all the barred relevant entities entering the field equations (6) are equal to the non-barred entities. Hence, if the field equations are satisfied for the non-barred entities, they are automatically satisfied for the barred entities.

Our solution (13) includes four terms and the solution derived in Ref. [1], Eq. (15), includes the *same four terms* plus the extra term $-\frac{r^2}{l^2} \sum_{i<j}^n (\omega_i d\phi_j - \omega_j d\phi_i)^2$, which in the case $N = 5, n = 2$ takes the form $-\frac{r^2}{l^2} (\omega_1 d\phi_2 - \omega_2 d\phi_1)^2$. It is clear that these two solutions are manifestly different. Even if they share some similar geometrical and physical properties they are certainly different solutions because *they cannot be related by a global coordinate transformation*.

4 Concluding remarks

We have thus shown that a trivial generalization of the transformation (8) to higher dimensional spacetimes is possible. By virtue of such a generalization we derived a simple cylindrically rotating solution of the form (5) with the last term $-(r^2/l^2) \sum_{i<j}^n (\omega_i d\phi_j - \omega_j d\phi_i)^2$ removed. This newly derived metric along with $A_\mu dx^\mu = \Phi(r)(\Xi dt - \sum_{i=1}^n \omega_i d\phi_i)$ is a solution to the field equations (6) provided the Minkowskian metric is diagonal $\eta_{ij} = \text{diag}(1, -1, -1, \dots, -1)$ with the tetrad given by the expression (12) of [1]. The r -functions (A, B, Φ) are given in (7).

Another, non-trivial, generalization of (8) is also possible yielding a *complicated* cylindrically rotating solution of the form (5). This metric along with $A_\mu dx^\mu = \Phi(r)(\Xi dt - \sum_{i=1}^n \omega_i d\phi_i)$ is a solution to the field equations (6) provided the Minkowskian metric is non-diagonal of the general form given in Eq. (44) of Ref. [8] and Eq. (41) of Ref. [9] with the tetrad given by the expression (12) of [1]. The r -functions (A, B, Φ) are given in (7).

We have also shown that if $(g_{\mu\nu}, \eta_{ij})$ is a rotating solution with η_{ij} being diagonal, then $(\bar{g}_{\mu\nu}, \bar{\eta}_{ij})$ is another rotating solution with $\bar{\eta}_{ij} = \eta_{ik} R_{kj}$ being non-diagonal and R_{ij} is

a symmetric matrix. These two rotating solutions have the same geometrical properties.

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors' comment: This is a theoretical study and no experimental data has been listed.]

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