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Coefficient bounds and distortion theorems for the certain analytic functions

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Abstract: In this paper, we introduce and investigate an analytic function class $P_q(\lambda, A, B)$ that we call the class of $q-$starlike and $q-$convex functions with respect to the parameter λ. We give coefficient bounds estimates, distortion bound and growth theorems for the functions belonging to this class.

Key words: Starlike function, convex function, $q-$starlike function, $q-$convex function, $q-$derivative

1. Introduction and preliminaries

Let $H(U)$ denote the class of analytic functions on the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ in the complex plane. Also, let T be the class of analytic functions $f \in H(U)$, which are normalized by $f(0) = 0 = f'(0) - 1$ with series expansion

$$f(z) = z - a_2 z^2 - a_3 z^3 - \cdots - a_n z^n - \cdots = z - \sum_{n=2}^{\infty} a_n z^n, a_n \geq 0. \tag{1.1}$$

Also, we denote by S the subclass of T consisting of functions which are also univalent in U.

We say that a function $f(z)$ is subordine to $g(z)$ and written as $f \prec g$, if there exists a (Schwartz) function $\omega(z)$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ such that $f(z) = g(\omega(z))$. In particular, when $g(z)$ is univalent then the above subordination is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$ (see [14]).

Some of the important subclasses of S are $S^*(\alpha)$ and $C(\alpha)$, respectively, starlike and convex functions of order $\alpha \geq 0$ (for details, see [11, 14] and [27]).

$$S^*(\alpha) = \left\{ f \in S : \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, z \in U \right\} \tag{1.2}$$

and

$$C(\alpha) = \left\{ f \in S : \Re\left(\frac{(zf'(z))'}{f'(z)}\right) > \alpha, z \in U \right\}. \tag{1.3}$$

$S^* = S^*(0)$ and $C = C(0)$ are, respectively, well-known starlike and convex functions in U. As it is known, geometrically, a function $f \in A$ is called starlike in U if $f(U)$ is starlike domain in the complex plane and convex in U if $f(U)$ is convex domain in the complex plane.

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Jackson [16] introduced the $q-$derivative operator D_q of a function $f \in H(U)$ as follows:

$$D_q f(z) = \begin{cases} \frac{f(z)-f(qz)}{(1-q)z}, & \text{if} \quad z \neq 0 \\ f^{'}(0), & \text{if} \quad z = 0 \end{cases}, \tag{1.4}$$

for $q \in (0,1)$. It is clear that $\lim_{q\to 1^-} D_q f(z) = f'(z)$ for $f \in H(U)$.

The following is the formula for the $q-$derivative of the function z^n as follows:

$$D_q z^n = [n]_q z^{n-1}, n \in \mathbb{N}, \tag{1.5}$$

where

$$[n]_q = \sum_{k=1}^{n} q^{k-1}, [0]_q = 0 \tag{1.6}$$

is the $q-$analogue of the natural numbers (which is called the basic number n). It follows from (1.6) that $[n]_q = (1-q^n)/(1-q)$ and $\lim_{q\to 1^-} [n]_q = n$ for $q \in (0,1)$.

Also,

$$D_q \left[f(z)g(z)\right] = f(qz)D_q g(z) + g(z)D_q f(z), \tag{1.7}$$

$$D_q \left[af(z) + bg(z)\right] = aD_q f(z) + bD_q g(z), a, b \in \mathbb{C}, \tag{1.8}$$

$$D_q \left[\frac{f(z)}{g(z)}\right] = \frac{g(z)D_q f(z) - f(z)D_q g(z)}{g(qz)g(z)}, g(qz)g(z) \neq 0 \tag{1.9}$$

and on a product of functions, the general Leibniz rule for action of powers of the $q-$derivative operator is

$$D_q^{(n)}(fg)(z) = \sum_{k=0}^{n} \binom{n}{k} D_q^{(k)} f(q^{n-k}z) D_q^{(n-k)} g(q^k z), \tag{1.10}$$

where $\binom{n}{k}_q$ is $q-$binomial coefficients given by

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q! \, [n-k]_q!},$$

for $k = 0, 1, 2, ..., n$ such that

$$[n]_q! = \prod_{k=1}^{n} [k]_q, [0]_q! = 1$$

is $q-$factorial function. Note that $\lim_{q\to 1^-} [n]_q! = n!$ is factorial function, and $\lim_{q\to 1^-} \binom{n}{k}_q = \binom{n}{k}$ are binomial coefficients. We refer to [12, 15, 18] about some properties of difference operator D_q.

It follows from (1.10) that $D_q^2 f(z) = D_q(D_q f(z))$, and $D_q^n f(z) = D_q(D_q^{n-1} f(z))$ for $n = 3, 4, ...$

It follows from (1.4) and (1.1) that

$$D_q f(z) = 1 - \sum_{n=2}^{\infty} [n]_q a_n z^{n-1} \tag{1.11}$$

and

$$D_q\left[zD_qf(z)\right]=D_qf(z)+zD_q^2f(z).$$

For $q\in(0,1)$, we define by $S_q^*(\alpha)$ and $C_q(\alpha)$ the subclass of T, which we will call, respectively, $q-starlike$ and $q-convex$ functions of order $\alpha\geq 0$, as follows

$$S_q^*(\alpha)=\left\{f\in S:\Re\frac{zD_qf(z)}{f(z)}>\alpha, z\in U\right\},$$

$$C_q(\alpha)=\left\{f\in S:\Re\left(\frac{D_q(zD_qf(z))}{D_qf(z)}\right)>\alpha, z\in U\right\}.$$

It is clear that $\lim\limits_{q\to1^-}S_q^*(\alpha)=S^*(\alpha)$ and $\lim\limits_{q\to1^-}C(\alpha)=C(\alpha)$.

Studies on $q-$difference equations were initiated by Jackson [17], Carmichael [10], Mason [19], Adams [1], and Trjizinsky [28]. A recent study in this subject was done by Bangerezako [9]. Some properties related with function theory involving $q-$theory were first introduced by Ismail et al. [15]. Moreover, there are more studies in this matter such as [2, 21, 23, 24]. As the study [2] suggests, there are a lot that can be done for this research topic. For example, $q-$analogy of starlikeness and convexity of analytic functions in the unit disk and in arbitrary simply connected domains would be interesting for researchers in this field. Recently, in [8], $q-$convexity for basic hypergeometric functions was studied. In a recent paper [24], Bieberbach conjecture problem for $q-$close-to-convex functions was estimated optimally. In fact, sharpness of their result is still open to debate as the authors also stated.

Recently, several subclasses of analytic functions have been investigated by many researchers using the $q-$derivative operator. For example, to define several subclasses of analytic functions and to study the extension of the theory of univalent functions, the theory of $q-$derivative, $q-$derivative operator can be used as tools.

In [3], by using applications of the $q-$derivative, it was shown that Szasz Mirakyan operators are convex when convex functions are taken such that their result generalizes well known results for $q=1$. Also, in [3], the authors showed that $q-$derivatives of these operators approach to $q-$derivatives of approximated functions. Recently, by Uçar et al. [30], some properties of $q-$close-to-convex functions were studied. In [22], Ramachandran et al. studied certain bound for $q-$starlike and $q-$convex functions with respect to symmetric points. In [22], the coefficient estimates for the subclasses of $q-$starlike and $q-$convex functions with respect to symmetric points involving $q-$difference operator were established and the authors provided certain applications based on these results for subclasses of univalent functions defined by convolution.

Very recently, $q-$starlike functions were investigated and basic characterization, growth theorem, and distortion theorem for this class were given by Polatoğlu [20]. Also, $q-$close-to-convex function with respect to Janowski starlike functions were studied and basic characterization, growth theorem, and distortion theorem for this class were given by Uçar [29].

Inspired by the studies mentioned above, we introduce the function class $P_q(\lambda,A,B)$ as follows.

Definition 1.1 *A function $f\in T$ given by (1.1) is said to be in the class $P_q(\lambda,A,B)$, $q\in(0,1)$, $\lambda\in[0,1]$ if the following condition is satisfied*

$$\frac{zD_qf(z)+\lambda z^2D_q^2f(z)}{\lambda zD_qf(z)+(1-\lambda)f(z)}\prec\frac{1+Az}{1+Bz}, z\in U,$$

where $-1 \leq B < A \leq 1$.

In the special case for $A = -B = 1$ and $A = 1 - 2\alpha,\ B = -1$ from the Definition 1.1, we can give the following definitions, respectively.

Definition 1.2 *A function* $f \in T$ *given by (1.1) is said to be in the class* $P_q(\lambda, 1, -1),\ q \in (0,1),\ \lambda \in [0,1]$ *if the following condition is satisfied*

$$\frac{zD_q f(z) + \lambda z^2 D_q^2 f(z)}{\lambda z D_q f(z) + (1-\lambda) f(z)} \prec \frac{1+z}{1-z}, z \in U.$$

Definition 1.3 *A function* $f \in T$ *given by (1.1) is said to be in the class* $P_q(\lambda, 1-2\alpha, -1),\ q \in (0,1),\ \lambda \in [0,1]$ *if the following condition is satisfied:*

$$\frac{zD_q f(z) + \lambda z^2 D_q^2 f(z)}{\lambda z D_q f(z) + (1-\lambda) f(z)} \prec \frac{1+(1-2\alpha)\, z}{1-z}, z \in U.$$

Note that classes $\lim\limits_{q\to 1^-} P_q(\lambda, A, B) = P(\lambda, A, B)$, and $P(\lambda, 1-2\alpha, -1)$ with $A = 1-2\alpha,\ B = -1$ were studied by Altıntas [4, 5] and Altıntas et al. [6, 7].

The main purpose of this paper is to give coefficient bounds estimates and distortion theorems for the functions belonging to the class $P_q(\lambda, A, B),\ q \in (0,1)$.

2. Coefficient bound estimates for the class $P_q(\lambda, A, B)$

In this section, we will investigate some inclusion results of the subclass $P_q(\lambda, A, B),\ q \in (0,1)$. Here, we provide coefficient estimates and some conditions for the functions belonging to this class.

A necessary condition for the functions belonging to the class $P_q(\lambda, A, B)$ is given by the following theorem.

Theorem 2.1 *If* $f \in T$ *belongs to the class* $P_q(\lambda, A, B),\ q \in (0,1),\ \lambda \in [0,1]$ *then the following condition is satisfied*

$$\sum_{n=2}^{\infty} \left([n]_q - \beta\right) \left[1 + \left([n]_q - 1\right)\lambda\right] a_n \leq 1 - \beta, \tag{2.1}$$

where $\beta = \frac{1-A}{1-B}, -1 \leq B < A \leq 1$. *The result obtained here is sharp.*

Proof Let $f \in T$. Throughout this study, we assume that $F(z)$ is defined by

$$F(z) = \lambda z D_q f(z) + (1-\lambda) f(z),\ z \in U. \tag{2.2}$$

It is clear that $F(z)$ is an analytic function in U with conditions

$$F(0) = 0 = D_q F(0) - 1.$$

A simple computation using (1.1), (1.11), and (2.2) shows that $F(z)$ has the following power series expansion

$$F(z) = z - \sum_{n=2}^{\infty} A_n z^n,\ z \in U, \tag{2.3}$$

where

$$A_n = \left[1 + \left([n]_q - 1\right)\lambda\right] a_n, \; n = 2, 3, ... \tag{2.4}$$

Assume that $f \in P_q(\lambda, A, B)$, $q \in (0,1)$, $\lambda \in [0,1]$, $-1 \leq B < A \leq 1$. Then,

$$\frac{zD_qF(z)}{F(z)} \prec \frac{1+Az}{1+Bz}, z \in U. \tag{2.5}$$

Using the definition of the subordination condition, we have

$$h(z) = \frac{zD_qF(z)}{F(z)} = \frac{1+A\omega(z)}{1+B\omega(z)}$$

with $\omega(0) = 0$ and $|\omega(z)| < 1$, $z \in U$.

It follows that

$$|\omega(z)| = \left|\frac{h(z)-1}{A - Bh(z)}\right| < 1.$$

Then, by taking $h(z) = u + iv$ from the last inequality, we find that

$$1 - A^2 + \left(1 - B^2\right)u^2 < 1 - A^2 + \left(1 - B^2\right)\left(u^2 + v^2\right) < 2u(1 - AB);$$

that is,

$$\left(1 - B^2\right)u^2 - 2u(1 - AB)u + \left(1 - A^2\right) < 0.$$

The last inequality implies that

$$\frac{1-A}{1-B} < u = \Re(h(z)) < \frac{1+A}{1+B}$$

(see also inequality (17) in [26]).

Thus, we write

$$\Re\left(\frac{zD_qF(z)}{F(z)}\right) > \beta, \tag{2.6}$$

where $\beta = \frac{1-A}{1-B}, -1 \leq B < A \leq 1$.

From (2.3) and (2.6) using (1.11) for the function $F(z)$, we obtain

$$\Re\left(\frac{z - \sum\limits_{n=2}^{\infty}[n]_q A_n z^n}{z - \sum\limits_{n=2}^{\infty} A_n z^n}\right) > \beta, \; z \in U.$$

The expression

$$\frac{z - \sum\limits_{n=2}^{\infty}[n]_q A_n z^n}{z - \sum\limits_{n=2}^{\infty} A_n z^n}$$

is real if the chosen z is real. Thus, from the last inequality letting $z \to 1$ through real values, we obtain

$$1-\sum_{n=2}^{\infty}[n]_q A_n \geq \beta\left(1-\sum_{n=2}^{\infty}A_n z^n\right),$$

which is equivalent to

$$\sum_{n=2}^{\infty}\left([n]_q-\beta\right)A_n \leq 1-\beta.$$

Taking values $A_n, n=2,3,...$ from (2.4), we complete the proof of inequality (2.1).

To see that result obtained in the theorem is sharp, we note that equality is attained in the inequality when $f(z)$ is chosen so that

$$f(z)=f_n(z)=z-\frac{1-\beta}{\left([n]_q-\beta\right)\left[1+\left([n]_q-1\right)\lambda\right]}z^n, z\in U, n=2,3,... \tag{2.7}$$

Hence, the result obtained in the theorem is sharp for the functions $f_n(z)$ defined by (2.7). Therefore, the proof of Theorem 2.1 is completed. □

From the Theorem 2.1, using different values of the parameters, we can easily obtain the following results.

Corollary 2.2 *If* $f\in T$ *belongs to the class* $P_q(\lambda,1,-1)$, $q\in(0,1)$, $\lambda\in[0,1]$ *then the following condition is satisfied:*

$$\sum_{n=2}^{\infty}\left[1+\left([n]_q-1\right)\lambda\right][n]_q a_n \leq 1.$$

The result obtained here is sharp.

Corollary 2.3 *If* $f\in T$ *belongs to the class* $P_q(\lambda,1-2\alpha,-1)$, $q\in(0,1)$, $\lambda\in[0,1]$, $\alpha\in[0,1)$, *then the following condition is satisfied:*

$$\sum_{n=2}^{\infty}\left([n]_q-\alpha\right)\left[1+\left([n]_q-1\right)\lambda\right]a_n \leq 1-\alpha.$$

The result obtained here is sharp.

Corollary 2.4 *If* $f\in T$ *belongs to the class* $S_q^*(\alpha)$, $q\in(0,1)$, $\alpha\in[0,1)$ *then the following condition is satisfied:*

$$\sum_{n=2}^{\infty}\left([n]_q-\alpha\right)a_n \leq 1-\alpha.$$

The result obtained here is sharp.

Corollary 2.5 *If $f \in T$ belongs to the class $C_q(\alpha)$, $q \in (0,1)$, $\alpha \in [0,1)$ then the following condition is satisfied:*

$$\sum_{n=2}^{\infty} \left([n]_q - \alpha \right) [n]_q a_n \leq 1 - \alpha.$$

The result obtained here is sharp.

Note that related results obtained in Corollaries 2.4 and 2.5 for the classes $S^*(\alpha)$ and $C(\alpha)$, respectively, were obtained by Silverman in [25] as follows.

Theorem 2.6 *(see [25], Theorem 2) A function $f \in T$ is in the class $S^*(\alpha)$, $\alpha \in [0,1)$ if and only if*

$$\sum_{n=2}^{\infty} (n - \alpha) a_n \leq 1 - \alpha.$$

Corollary 2.7 *(see [25], Corollary 2) A function $f \in T$ is in the class $S^*(\alpha)$, $\alpha \in [0,1)$ if and only if*

$$\sum_{n=2}^{\infty} n (n - \alpha) a_n \leq 1 - \alpha.$$

On the coefficient bounds of the functions belonging to the class $P_q(\lambda, A, B)$, we give the following theorem.

Theorem 2.8 *If $f \in T$ belongs to the class $P_q(\lambda, A, B)$, $q \in (0,1)$, $\lambda \in [0,1]$ then the following condition is satisfied:*

$$\begin{aligned} \sum_{n=2}^{\infty} a_n &\leq \frac{1-\beta}{\left([n]_q - \beta \right) \left[1 + \left([n]_q - 1 \right) \lambda \right]}, \\ \sum_{n=2}^{\infty} [n]_q a_n &\leq \frac{(1-\beta) [n]_q}{\left([n]_q - \beta \right) \left[1 + \left([n]_q - 1 \right) \lambda \right]}, n = 2, 3, ..., \end{aligned}$$

where $\beta = \frac{1-A}{1-B}, -1 \leq B < A \leq 1$. The results obtained here are sharp.

Proof Let $f \in P_q(\lambda, A, B)$, $q \in (0,1)$, $\lambda \in [0,1]$. Then, using Theorem 2.1, we can write

$$\left([n]_q - \beta \right) \left[1 + \left([n]_q - 1 \right) \lambda \right] \sum_{n=2}^{\infty} a_n \leq \sum_{n=2}^{\infty} \left([n]_q - \beta \right) \left[1 + \left([n]_q - 1 \right) \lambda \right] a_n \leq 1 - \beta,$$

which is equivalent to the first assertion of the theorem.

Similarly, we write

$$\left[1 + \left([n]_q - 1 \right) \lambda \right] \sum_{n=2}^{\infty} \left([n]_q - \beta \right) a_n \leq \sum_{n=2}^{\infty} \left([n]_q - \beta \right) \left[1 + \left([n]_q - 1 \right) \lambda \right] a_n \leq 1 - \beta;$$

that is,

$$\left[1+\left([n]_q-1\right)\lambda\right]\sum_{n=2}^{\infty}[n]_q\,a_n\leq 1-\beta+\left[1+\left([n]_q-1\right)\lambda\right]\beta\sum_{n=2}^{\infty}a_n,$$

In the right side of the last inequality using the first assertion of the theorem, we arrive at the following inequality

$$\left[1+\left([n]_q-1\right)\lambda\right]\sum_{n=2}^{\infty}[n]_q\,a_n\leq\frac{(1-\beta)\,[n]_q}{[n]_q-\beta}.$$

From here, we can easily see that the second assertion of the theorem is true.

It is clear that the results obtained in theorem are sharp for the functions defined by (2.7).

Thus, the proof of Theorem 2.8 is completed. □

Remark 2.9 *Numerous consequences of the results obtained in Theorem 2.8 can be given for different values of parameters* $A, B, \lambda,$ *and* β.

The following theorem is direct result of Theorem 2.1.

Theorem 2.10 *If* $f\in T$ *belongs to the class* $P_q(\lambda,A,B),$ $q\in(0,1),$ $\lambda\in[0,1]$ *then the following inequality for the coefficients is satisfied:*

$$a_n\leq\frac{1-\beta}{\left([n]_q-\beta\right)\left[1+\left([n]_q-1\right)\lambda\right]},\ n=2,3,...,$$

where $\beta=\frac{1-A}{1-B}, -1\leq B<A\leq 1.$

From Theorem 2.10, we arrive at the following results.

Corollary 2.11 *If* $f\in T$ *belongs to the class* $P_q(\lambda,1,-1),$ $q\in(0,1),$
$\lambda\in[0,1]$ *then the following inequality for the coefficients is satisfied:*

$$a_n\leq\frac{1}{\left[1+\left([n]_q-1\right)\lambda\right][n]_q},\ n=2,3,...$$

Corollary 2.12 *If* $f\in T$ *belongs to the class* $P_q(\lambda,1-2\alpha,-1),$ $q\in(0,1),$ $\lambda\in[0,1],$ $\alpha\in[0,1)$ *then the following inequality for the coefficients is satisfied:*

$$a_n\leq\frac{1-\alpha}{\left([n]_q-\alpha\right)\left[1+\left([n]_q-1\right)\lambda\right]},\ n=2,3,...$$

Corollary 2.13 *If* $f\in T$ *belongs to the class* $S_q^*(\alpha),$ $q\in(0,1),$ $\alpha\in[0,1)$ *then the following inequality for the coefficients is satisfied:*

$$a_n\leq\frac{1-\alpha}{[n]_q-\alpha},\ n=2,3,...$$

Corollary 2.14 *If $f \in T$ belongs to the class $C_q(\alpha)$, $q \in (0,1)$, $\alpha \in [0,1)$ then the following inequality for the coefficients is satisfied:*

$$a_n \leq \frac{1-\alpha}{\left([n]_q - \alpha\right)[n]_q}, \; n = 2, 3, ...$$

Remark 2.15 *Numerous consequences of the results obtained in the above corollaries can be given for different values of parameters λ and α.*

3. Distortion bound and growth theorems for the class $P_q(\lambda, A, B)$

In this section, we give distortion and growth theorems for the functions belonging to the class $P_q(\lambda, A, B)$.

Our coefficient bounds enable us to prove the following theorem.

Theorem 3.1 *Let $f \in T$ belong to the class $P_q(\lambda, A, B)$, $q \in (0,1)$, $\lambda \in [0,1]$. Then,*

$$\begin{aligned} r - \frac{1-\beta}{\left([2]_q - \alpha\right)\left[1 + \left([2]_q - 1\right)\lambda\right]} r^2 \quad &\leq \quad |f(z)| \leq r + \frac{1-\beta}{\left([2]_q - \alpha\right)\left[1 + \left([2]_q - 1\right)\lambda\right]} r^2, \\ |z| \quad &= \quad r, \; r \leq 1, \end{aligned}$$

where $\beta = \frac{1-A}{1-B}, -1 \leq B < A \leq 1$, with equality for the function

$$f(z) = z - \frac{1-\beta}{\left([2]_q - \beta\right)\left[1 + \left([2]_q - 1\right)\lambda\right]} z^2, |z| = r. \tag{3.1}$$

Proof Assume that $f \in P_q(\lambda, A, B)$, $q \in (0,1)$, $\lambda \in [0,1]$. Then, using Theorem 2.1, we can write

$$\left([2]_q - \beta\right)\left[1 + \left([2]_q - 1\right)\lambda\right] \sum_{n=2}^{\infty} a_n \leq \sum_{n=2}^{\infty} \left([n]_q - \beta\right)\left[1 + \left([n]_q - 1\right)\lambda\right] a_n < 1 - \beta;$$

that is,

$$\sum_{n=2}^{\infty} a_n \leq \frac{1-\beta}{\left([2]_q - \beta\right)\left[1 + \left([2]_q - 1\right)\lambda\right]},$$

Using this, we can easily show that

$$|f(z)| \leq r + \sum_{n=2}^{\infty} a_n r^n \leq r + r^2 \sum_{n=2}^{\infty} a_n \leq r + \frac{1-\beta}{\left([2]_q - \beta\right)\left[1 + \left([2]_q - 1\right)\lambda\right]} r^2. \tag{3.2}$$

On the other hand,

$$|f(z)| \geq r - \sum_{n=2}^{\infty} a_n r^n \geq r - r^2 \sum_{n=2}^{\infty} a_n \geq r - \frac{1-\beta}{\left([2]_q - \beta\right)\left[1 + \left([2]_q - 1\right)\lambda\right]} r^2. \tag{3.3}$$

Thus, the combination of (3.2) and (3.3) completes the proof of the inequalities.

It is clear that equalities are attained in the inequalities when the function $f(z)$ is chosen as in (3.1). Thus, the proof of Theorem 3.1 is completed. □

From the Theorem 3.1, we arrive at the following results.

Corollary 3.2 *Let $f \in T$ belong to the class $P_q(\lambda, 1, -1)$, $q \in (0,1)$, $\lambda \in [0,1]$. Then,*

$$r - \frac{1}{\left[1+\left([2]_q - 1\right)\lambda\right][2]_q} r^2 \le |f(z)| \le r + \frac{1}{\left[1+\left([2]_q - 1\right)\lambda\right][2]_q} r^2, |z| = r, \ r \le 1,$$

such that equality is achieved for the function

$$f(z) = z - \frac{1}{\left[1+\left([2]_q - 1\right)\lambda\right][2]_q} z^2, |z| = r.$$

Corollary 3.3 *Let $f \in T$ belong to the class $P_q(\lambda, 1-2\alpha, -1)$, $q \in (0,1)$, $\lambda \in [0,1]$, $\alpha [0,1)$. Then,*

$$\begin{aligned} r - \frac{1-\alpha}{\left([2]_q - \alpha\right)\left[1+\left([2]_q - 1\right)\lambda\right]} r^2 &\le |f(z)| \le r + \frac{1-\alpha}{\left([2]_q - \alpha\right)\left[1+\left([2]_q - 1\right)\lambda\right]} r^2, \\ |z| &= r, \ r \le 1 \end{aligned}$$

such that equality is achieved for the function

$$f(z) = z - \frac{1-\alpha}{\left([2]_q - \alpha\right)\left[1+\left([2]_q - 1\right)\lambda\right]} z^2, |z| = r.$$

Corollary 3.4 *Let $f \in T$ belong to the class $S_q^*(\alpha)$, $q \in (0,1)$, $\alpha \in [0,1)$. Then,*

$$r - \frac{1-\alpha}{[2]_q - \alpha} r^2 \le |f(z)| \le r + \frac{1-\alpha}{[2]_q - \alpha} r^2, |z| = r, \ r \le 1,$$

such that equality is achieved for the function

$$f(z) = z - \frac{1-\alpha}{[2]_q - \alpha} z^2, |z| = r.$$

Corollary 3.5 *Let $f \in T$ belong to the class $C_q(\alpha)$, $q \in (0,1)$, $\alpha \in [0,1)$. Then,*

$$r - \frac{1-\alpha}{\left([2]_q - \alpha\right)[2]_q} r^2 \le |f(z)| \le r + \frac{1-\alpha}{\left([2]_q - \alpha\right)[2]_q} r^2, |z| = r, \ r \le 1,$$

such that equality is achieved for the function

$$f(z) = z - \frac{1-\alpha}{\left([2]_q - \alpha\right)[2]_q} z^2, |z| = r.$$

Remark 3.6 *Numerous consequences of the results obtained in the above corollaries can be given for different values of parameters λ and α.*

The related results obtained in Corollaries 3.4 and 3.5 for the classes $S^*(\alpha)$ and $C(\alpha)$, respectively, were obtained by Silverman [25] as follows.

Theorem 3.7 *(see [25], Theorem 4) If $f \in S^*(\alpha)$, $\alpha \in [0,1)$, then*

$$r - \frac{1-\alpha}{2-\alpha} r^2 \leq |f(z)| \leq r + \frac{1-\alpha}{2-\alpha} r^2, |z| = r, \ r \leq 1$$

such that equality is achieved for the function

$$f(z) = z - \frac{1-\alpha}{2-\alpha} z^2, |z| = r.$$

Corollary 3.8 *(see [25], Corollary of the Theorem 4) If $f \in C(\alpha)$, $\alpha \in [0,1)$, then*

$$r - \frac{1-\alpha}{2(2-\alpha)} r^2 \leq |f(z)| \leq r + \frac{1-\alpha}{2(2-\alpha)} r^2, |z| = r, \ r \leq 1$$

such that equality is achieved for the function

$$f(z) = z - \frac{1-\alpha}{2(2-\alpha)} z^2, |z| = r.$$

The following theorem is the growth theorem for the functions belonging to class $P_q(\lambda, A, B)$.

Theorem 3.9 *Let $f \in T$ belong to the class $P_q(\lambda, A, B)$, $q \in (0,1)$, $\lambda \in [0,1]$. Then,*

$$\begin{aligned} 1 - \frac{(1-\beta)[2]_q}{\left([2]_q - \beta\right)\left[1 + \left([2]_q - 1\right)\lambda\right]} r &\leq |D_q f(z)| \leq 1 + \frac{(1-\beta)[2]_q}{\left([2]_q - \beta\right)\left[1 + \left([2]_q - 1\right)\lambda\right]} r, \\ |z| &= r, \ r \leq 1, \end{aligned}$$

where $\beta = \frac{1-A}{1-B}, -1 \leq B < A \leq 1$, such that equality is achieved for the function

$$f(z) = z - \frac{(1-\beta)[2]_q}{\left([2]_q - \beta\right)\left[1 + \left([2]_q - 1\right)\lambda\right]} z^2, |z| = r. \tag{3.4}$$

Proof Let $f \in P_q(\lambda, A, B)$, $q \in (0,1)$, $\lambda \in [0,1]$. By simple computation, we obtain

$$|D_q f(z)| \leq 1 + \sum_{n=2}^{\infty} [n]_q a_n \left|z^{n-1}\right| \leq 1 + r \sum_{n=2}^{\infty} [n]_q a_n.$$

Using the second assertion of the Theorem 2.8, we obtain

$$|D_q f(z)| \leq 1 + \frac{(1-\beta)[2]_q}{\left([2]_q - \beta\right)\left[1 + \left([2]_q - 1\right)\lambda\right]} r.$$

Similarly, we can easily show that

$$|D_q f(z)| \geq 1 - \sum_{n=2}^{\infty} [n]_q \, a_n \left| z^{n-1} \right| \geq 1 - \frac{(1-\beta)\,[2]_q}{\left([2]_q - \beta\right)\left[1 + \left([2]_q - 1\right)\lambda\right]} r.$$

Thus, the proof of the inequalities is completed.

We can easily show that equalities are attained in the inequalities when the function $f(z)$ is chosen as in (3.4).

With this, the proof of Theorem 3.9 is completed. □

Remark 3.10 *Numerous consequences of the results obtained in Theorem 3.9 can be given for different values of parameters A, B and λ .*

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