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Parametrizing numerical semigroups with multiplicity up to 5

Halil İbrahim Karakaş
Faculty of Commercial Sciences
Başkent University, Ankara, Turkey
karakas@baskent.edu.tr

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In this work, we give parametrizations in terms of the Kunz coordinates of numerical semigroups with multiplicity up to 5. We also obtain parametrizations of MED semigroups, symmetric and pseudo-symmetric numerical semigroups with multiplicity up to 5. These parametrizations also lead to formulas for the number of numerical semigroups, the number of MED semigroups and the number of symmetric and pseudo-symmetric numerical semigroups with multiplicity up to 5 and given conductor.

Keywords: Numerical semigroups; embedding dimension; multiplicity; conductor; Frobenius number; genus; Apéry sets; MED semigroups; symmetric and pseudo-symmetric numerical semigroups.

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1. Introduction

Let $\mathbb{N}$ denote the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, the set of non-negative integers. A subset $S \subseteq \mathbb{N}_0$ satisfying

$$\text{(i) } 0 \in S \quad \text{(ii) } S + S \subseteq S \quad \text{(iii) } |\mathbb{N}_0 \backslash S| < \infty$$

is called a *numerical semigroup* ($K + K = \{x + y : x, y \in K\}$ and $|K|$ denotes the cardinality of K for any set K of integers). It is well known (see, for instance, [1, 4, 13]) that the condition (iii) above is equivalent to saying that the greatest common divisor $\gcd(S)$ of elements of S is 1.

The smallest integer $c = c(S) \in S$ with $\{c\} + \mathbb{N}_0 \subseteq S$ is called the *conductor* of S. Then $c - 1 = \mathsf{f} = \mathsf{f}(S)$ is the largest integer that is not an element of S and it is called the *Frobenius number* of S. Clearly, $c(\mathbb{N}_0) = 0$, $\mathsf{f}(\mathbb{N}_0) = -1$ and $c(S) > 1$ if and only if $S \neq \mathbb{N}_0$.

Those positive integers not belonging to S are called *gaps* of S. The number of gaps of a numerical semigroup S is called the *genus* of S and it is denoted by $g = g(S)$. The largest gap of S is $\mathsf{f}(S)$ if $S \neq \mathbb{N}_0$.

If A is a subset of $\mathbb{N}_0$, we will denote by $\langle A \rangle$ the submonoid of $\mathbb{N}_0$ generated by A. The monoid $\langle A \rangle$ is a numerical semigroup if and only if $\gcd(A) = 1$. If $A = \{a_1, \ldots, a_n\}$, we write $\langle A \rangle = \langle a_1, \ldots, a_n \rangle$.

Every numerical semigroup S admits a unique *minimal system of generators* $\{a_1, a_2, \ldots, a_e\}$ with $a_1 < a_2 < \cdots < a_e$; that is, $\{a_1, a_2, \ldots, a_e\}$ generates S but no proper subset of it generates S. It is known that the minimal system of generators of S is $S^* \backslash (S^* + S^*)$, where $K^* = K \backslash \{0\}$ for any set K of integers (see, for instance, [13]). The cardinality of the minimal system of generators of a numerical semigroup S is called the *embedding dimension* of S, denoted $e = e(S)$. The smallest positive element of S is called the *multiplicity* of S and it is denoted by $m = m(S)$. It is well known that $e(S) \leq m(S)$ (see, for instance, [1] or [13]). A numerical semigroup is said to have maximal embedding dimension if $e(S) = m(S)$. Such semigroups are called, in short, MED semigroups (see [15]).

A numerical semigroup S is said to be *symmetric* if its Frobenius number $\mathsf{f}(S)$ is odd and $x \in \mathbb{Z} \backslash S$ implies $\mathsf{f}(S) - x \in S$. S is said to be *pseudo-symmetric* if its Frobenius number $\mathsf{f}(S)$ is even and $x \in \mathbb{Z} \backslash S$ implies either $\mathsf{f}(S) - x \in S$ or $x = \frac{\mathsf{f}(S)}{2}$. There are many equivalent definitions for symmetric and pseudo-symmetric numerical semigroups in the literature. Lemma 2.4 below gives a characterization of symmetric and pseudo-symmetric numerical semigroups (see, for instance, [4] or [13]).

For any integers z and m with $m > 1$, let $\overline{z} \in \{0, 1, \ldots, m-1\}$ such that $z \equiv \overline{z} \pmod{m}$.

If S is a numerical semigroup with multiplicity $m > 1$ and conductor c, then $c - 1 \notin S$ and therefore $c \not\equiv 1 \pmod{m}$.

Thus, if c is the conductor of a numerical semigroup with multiplicity $m > 1$, then $c \equiv \overline{c} \pmod{m}$, where $\overline{c} \in \{0, 2, \ldots, m-1\}$. Throughout this paper, we will use the notation

$$c = mb + \overline{c} \quad \text{with } b \in \mathbb{N} \quad \text{and} \quad \overline{c} \in \{0, 2, \ldots, m-1\}.$$

Numerical semigroups with multiplicity 3 and 4 are studied by many authors. It is shown in [12] that a numerical semigroup with multiplicity 3 is uniquely determined by its Frobenius number and genus; a numerical semigroup with multiplicity 4 is uniquely determined by its Frobenius number, genus and ratio (which is the least element of the minimal system of generators greater than the multiplicity). In [12], a formula is given for the number of numerical semigroups with multiplicity 3 and Frobenius number $\mathsf{f}(S)$. Formulas for the number of numerical semigroups and MED semigroups with multiplicity 3 and 4 and given Frobenius number are given in [2], too. In [7], formulas are given for the number of numerical semigroups with certain multiplicity and genus. There are also GAP packages that can be used to compute the number of numerical semigroups in classes mentioned above and the like (see for instance [6] or [3]). In what follows, we give parametrizations of numerical semigroups (together with MED semigroups, symmetric and pseudo-symmetric numerical semigroups) with multiplicity up to 5 and given conductor. These parametrizations also lead to formulas for the number of numerical semigroups (also MED semigroups, symmetric and pseudo-symmetric numerical semigroups) with multiplicity up to 5 and given conductor. We prefer to work with the conductor instead of the Frobenius number, perhaps because the conductor is an element of the semigroup.

We shall denote the number of numerical semigroups with multiplicity m and conductor c by $N(m,c)$; the number of MED semigroups with multiplicity m and conductor c by $N_{\mathrm{MED}}(m,c)$. Similarly, the number of symmetric, and pseudo-symmetric numerical semigroups with multiplicity m and conductor c will be denoted by $N_{\mathrm{SYM}}(m,c)$, and $N_{\mathrm{PSYM}}(m,c)$, respectively.

2. Apéry Sets and Kunz Coordinates

Let S be a numerical semigroup, $a \in S^*$. We define the *Apéry set* of S with respect to a, denoted $\mathrm{Ap}(S,a)$, to be the set

$$\mathrm{Ap}(S,a) = \{s \in S : s - a \notin S\}.$$

It is well known (see, for instance, [1, 13]) that

$$\mathrm{Ap}(S,a) = \{0 = w(0), w(1), \ldots, w(a-1)\},$$

where $w(i) = \min\{s \in S : s \equiv i \pmod{a}\}$ for each $i \in \{1, \ldots, a-1\}$.

Observe that for every integer z, there exist a unique integer $i \in \{0, 1, \ldots, a-1\}$ and a unique integer k such that $z = w(i) + ka$. Moreover, $z \in S \Leftrightarrow k \geq 0$. It follows from this observation that S is generated by $\{a\} \cup \mathrm{Ap}(S,a)^*$ for any $a \in S^*$. The formulas below are known as Selmer's formulas (see [13, Proposition 2.12]).

$$\mathsf{f} = \max(\mathrm{Ap}(S,a)) - a, \quad g(S) = \frac{1}{a}(w(1) + \cdots + w(a-1)) - \frac{a-1}{2}. \tag{1}$$

The following properties characterize $\mathrm{Ap}(S,a)$: For any $i, j \in \{0, 1, \ldots, a-1\}$, we have

$$w(i) \equiv w(j) \pmod{a} \;\Leftrightarrow\; i = j, \tag{2}$$

$$w(i) + w(j) = w(\overline{i+j}) + ta \quad \text{for some } t \in \mathbb{N}_0. \tag{3}$$

Since every numerical semigroup is generated by its multiplicity m together with nonzero elements of its Apéry set with respect to m, all properties of a numerical semigroup should have interpretations in terms of its Apéry set. This fact has already been used by many authors and several characterizations have been given for numerical semigroups in general and different classes of numerical semigroups in particular. Kunz [8] has used the Apéry sets of a numerical semigroup to define the so called Kunz coordinates which characterize the semigroup completely. Kunz and

Waldi [9], Blanco *et al.* [2], and Kaplan [7] used the Apéry sets and the Kunz coordinates to find formulas for the number of elements in certain classes of numerical semigroups.

The Apéry set $\mathrm{Ap}(S,m) = \{w(0) = 0, w(1), \ldots, w(m-1)\}$ of S with respect to its multiplicity m is the smallest Apéry set of S and each non-negative element of $\mathrm{Ap}(S,m)$ is greater than m. In other words, there exists a unique $k_i \in \mathbb{N}$, called *the ith Kunz coordinate* of S, such that

$$w(i) = k_i m + i$$

for each $i \in \{1, 2, \ldots, m-1\}$. The vector $\kappa = (k_1, k_2, \ldots, k_{m-1})$ is called the *Kunz vector* of S (see [8, 9, 14]).

The second identity in (1) can be expressed in terms of the Kunz coordinates as

$$g(S) = k_1 + k_2 + \cdots + k_{m-1}.$$

Using the identities in (3), one can see that for $i, j \in \{1, 2, \ldots, m-1\}$,

$$k_i + k_j - k_{\overline{i+j}} \geq \begin{cases} 0 & \text{if } 1 \leq i+j < m, \\ -1 & \text{if } i+j > m. \end{cases} \tag{4}$$

The next lemma which can be derived from [11, Lemma 3.3] or [14, Theorem 11], shows that the inequalities in (4) are sufficient for a vector $\kappa = (k_1, \ldots, k_{m-1})$ with components in $\mathbb{N}$ to be the Kunz vector of a numerical semigroup with multiplicity m.

Lemma 2.1. *Let $m \in \mathbb{N}, m > 1$, and let $\kappa = (k_1, \ldots, k_{m-1}) \in \mathbb{N}^{m-1}$. Then κ is the Kunz vector of a numerical semigroup S with multiplicity m if and only if the inequalities in* (4) *are satisfied, and when that is the case*

$$\mathrm{Ap}(S,m) = \{0, k_1 m + 1, \ldots, k_{m-1} m + m - 1\}.$$

Lemma 2.1 implies that a numerical semigroup is completely determined by its Kunz vector. Hence to determine a numerical semigroup with multiplicity m, it suffices to determine the corresponding vector $\kappa = (k_1, \ldots, k_{m-1}) \in \mathbb{N}^{m-1}$ satisfying the inequalities in (4). The vectors satisfying the inequalities in (4) constitute the coordinates of the polytope defined by those inequalities in $\mathbb{N}^{m-1}$.

Let S be a numerical semigroup with multiplicity m and conductor c. We have $\mathsf{f} = \max(\mathrm{Ap}(S,m)) - m$ by (1). Hence

$$c = \max(\mathrm{Ap}(S,m)) - m + 1.$$

We set $\mathcal{M} = \max(\mathrm{Ap}(S,m))$ and call $\mathcal{M}$ the *major* of S. Thus, $\mathcal{M} = c + m - 1$. If $\kappa = (k_1, k_2, \ldots, k_{m-1})$ is the Kunz vector of S, then there exists $i^* \in \{1, \ldots, m-1\}$ such that

$$\mathcal{M} = w(i^*) = k_{i^*} m + i^*.$$

We call k_{i*} the *major Kunz coordinate* of S. It is easily seen that

$$\mathcal{M} = \begin{cases} w(m-1) & \text{if } \overline{c} = 0, \\ w(\overline{c}-1) & \text{if } \overline{c} \neq 0 \end{cases}$$

and thus

$$k_{i^*} = \begin{cases} k_{m-1} = \dfrac{c}{m} & \text{if } \overline{c} = 0, \\ k_{\overline{c}-1} = \dfrac{c - \overline{c}}{m} + 1 & \text{if } \overline{c} \neq 0. \end{cases} \tag{5}$$

Let us also note that for any $j \in \{1, \ldots, m-1\}$, we have

$$\begin{aligned} \mathcal{M} + w(j) &= (k_{i^*} m + i^*) + (k_j m + j) \\ &= (k_{i*} + k_j)m + (i^* + j) \\ &\geq (k_{i^*} + 1)m + (i^* + j) \\ &\geq (k_{\overline{i^*+j}} + 1)m + (i^* + j) \end{aligned}$$

which implies

$$(k_{i*} + k_j) - k_{\overline{i^*+j}} \geq 1.$$

Hence, the inequalities (4) in Lemma 2.1 are satisfied if $k_i = k_{i*}$ or $k_j = k_{i*}$. Then Lemma 2.1 can be refined as follows.

Lemma 2.2. *Let $m \in \mathbb{N},\ m > 1,$ and let $\kappa = (k_1, \ldots, k_{m-1}) \in \mathbb{N}^{m-1}$. Assume that*

$$\max\{k_i + i : 1 \leq i \leq m-1\} = k_{i^*} + i^*.$$

Then κ is the Kunz vector of a numerical semigroup with multiplicity m and major $\mathcal{M} = k_{i^*} + i^*$ if and only if

$$k_i + k_j - k_{\overline{i+j}} \geq \begin{cases} 0 & \text{if } 1 \leq i + j < m, \\ -1 & \text{if } i + j > m \end{cases}$$

for all $i, j \in \{1, \ldots, m-1\} \backslash \{i^*\}$.

Lemma 2.2 shows that numerical semigroups with given multiplicity and major (or conductor) are in one to one correspondence with lattice points of the polytope defined by the inequalities in (4).

MED semigroups also can be characterized by their Kunz vectors. Theorem 15 of [14] and the observations just before Lemma 2.2 above can be used to prove the following lemma.

Lemma 2.3. *Let $m \in \mathbb{N}, m > 1,$ and let $\kappa = (k_1, \ldots, k_{m-1}) \in \mathbb{N}^{m-1}$. Assume that*

$$\max\{k_i + i : 1 \leq i \leq m-1\} = k_{i^*} + i^*.$$

Then κ is the Kunz vector of a MED *semigroup with multiplicity m and major $\mathcal{M} = k_{i^*} + i^*$ if and only if*

$$k_i + k_j - k_{\overline{i+j}} \geq \begin{cases} 1 & \text{if } 1 \leq i+j < m, \\ 0 & \text{if } i+j > m \end{cases}$$

for all $i, j \in \{1, \ldots, m-1\} \backslash \{i^*\}$.

The following lemma, which can be deduced from [13, Proposition 2.12 and Corollary 4.5], shows that the Kunz coordinates of symmetric or pseudo-symmetric numerical semigroups of given multiplicity and conductor correspond to the lattice points lying on a hyperplane of the polytope mentioned above.

Lemma 2.4. *Let $m, c \in \mathbb{N}$, $c \geq m > 1$ and let S be a numerical semigroup with multiplicity m, conductor c and Kunz vector $\kappa = (k_1, \ldots, k_{m-1}) \in N^{m-1}$. Then*

(i) *S is symmetric if and only if c is even and $k_1 + \cdots + k_{m-1} = \frac{c}{2}$,*
(ii) *S is pseudo-symmetric if and only if c is odd and $k_1 + \cdots + k_{m-1} = \frac{c+1}{2}$.*

We observe that there is no symmetric numerical semigroup whose conductor is odd, and there is no pseudo-symmetric numerical group whose conductor is even.

3. Numerical Semigroups with Low Multiplicity

In this section, we focus on parametrization of numerical semigroups with multiplicity up to 5 and given conductor. Let us note once more that if c is the conductor of a numerical semigroup with multiplicity m, then $c \not\equiv 1 \pmod{m}$. We shall use the notations

$$c = mb + \overline{c} \quad \text{with } b \in \mathbb{N} \quad \text{and} \quad \overline{c} \in \{0, 2, \ldots, m-1\}.$$

3.1. *Numerical semigroups with multiplicity* 1

The only numerical semigroup with multiplicity 1 is $\mathbb{N}_0$ which is a MED semigroup, and which is also symmetric.

3.2. *Numerical semigroups with multiplicity* 2

If S is a numerical semigroup with multiplicity 2 and conductor c, then c is even and $S = \langle 2, c+1 \rangle$. We see that a numerical semigroup with multiplicity 2 is completely determined by its conductor and that any numerical semigroup with multiplicity 2 is a MED semigroup. Any numerical semigroup with multiplicity 2 is symmetric by Lemma 2.4.

Numerical semigroups with multiplicity 3 or more are not completely determined by their conductor alone. Rosales [12] proves that a numerical semigroup with multiplicity 3 is completely determined by its genus and Frobenius number (or conductor). Moreover, a formula is given in that paper for the number of numerical semigroups with multiplicity 3 and Frobenius number f. Blanco *et al.* [2] also contains formulas for the number of numerical semigroups with multiplicity 3 and given Frobenius number as well as a formula for the number of numerical semigroups with multiplicity 4 and given Frobenius number; the latter being obtained by computer facilities. In [10], parametrizations are given for almost symmetric numerical semigroups with multiplicity 5 and embedding dimension 4; and parametrizations are given for Arf numerical semigroups with given conductor and multiplicity up to 7 in [5]. We give below parametrizations of numerical semigroups (MED semigroups, symmetric and pseudo-symmetric numerical semigroups) with given conductor and multiplicity 3, 4 or 5.

Given a rational number q, we shall use the customary notation $\lceil q \rceil$ for the smallest integer which is not less than q, and $\lfloor q \rfloor$ for the largest integer which is not greater than q.

3.3. *Numerical semigroups with multiplicity* 3

The following proposition describes all numerical semigroups with multiplicity 3 and conductor c.

Proposition 3.1. *Let S be a numerical semigroup with multiplicity* 3 *and conductor $c = 3b + \overline{c}, b \in \mathbb{N}$.*

(i) *If $\overline{c} = 0$, then $S = \langle 3, 3k_1 + 1, 3b + 2\rangle$, where $\lceil \frac{b}{2} \rceil \leq k_1 \leq b$,*
(ii) *If $\overline{c} = 2$, then $S = \langle 3, 3b + 4, 3k_2 + 2\rangle$, where $\lceil \frac{b}{2} \rceil \leq k_2 \leq b$.*

Proof. Assume that S is a numerical semigroup with multiplicity 3 and conductor $c = 3b + \overline{c}$. Then $\mathcal{M} = c + 2$. Let the Kunz vector of S be (k_1, k_2).

(i) If $\overline{c} = 0$, then the major Kunz coordinate of S is $k_2 = b$ by (5). Therefore, $k_1 \leq b$ and $2k_1 - b \geq 0$, i.e. $k_1 \geq \frac{b}{2}$ by Lemma 2.2. This proves the first assertion.
(ii) If $\overline{c} = 2$, then the major Kunz coordinate of S is $k_1 = b + 1$ by (5). Therefore, $k_2 \leq b$ and $2k_2 - (b + 1) \geq -1$, i.e. $k_2 \geq \frac{b}{2}$ by Lemma 2.2. This proves the second assertion. □

Proposition 2.1 can be used to calculate the number $N(3, c)$.

Corollary 3.2. *If $c \not\equiv 1 \pmod 3$, then the number of numerical semigroups with multiplicity* 3 *and conductor c is $N(3, c) = \frac{c-\overline{c}}{3} - \lceil \frac{c-\overline{c}}{6} \rceil + 1$.*

Corollary 8 of [12] and Proposition 4.3 of [2] give formulas for $N(3, c)$ in terms of the Frobenius number. The formula in [2] is exactly the same as the above formula, but the formula given in [12] is different.

Using Lemma 2.3 instead of Lemma 2.2 in the above proof, we obtain the following results.

Corollary 3.3. *Let S be a MED semigroup with multiplicity* 3 *and conductor* $c = 3b + \overline{c}, b \in \mathbb{N}$.

(i) *If* $\overline{c} = 0$, *then* $S = \langle 3, 3k_1 + 1, 3b + 2\rangle$, *where* $\lceil \frac{b+1}{2} \rceil \le k_1 \le b$,
(ii) *If* $\overline{c} = 2$, *then* $S = \langle 3, 3b + 4, 3k_2 + 2\rangle$, *where* $\lceil \frac{b+1}{2} \rceil \le k_2 \le b$.

Corollary 3.4. *If* $c \not\equiv 1 \pmod 3$, *then the number of MED semigroups with multiplicity* 3 *and conductor* c *is* $N_{\mathrm{MED}}(3, c) = \frac{c-\overline{c}}{3} - \lceil \frac{c-\overline{c}+3}{6} \rceil + 1$.

There is a similar formula for $N_{\mathrm{MED}}(3, c)$ in [2] in terms of the Frobenius number.

Proposition 2.1 together with Lemma 2.4 can be used to characterize symmetric and pseudo-symmetric numerical semigroups with multiplicity 3.

Corollary 3.5. *Assume that* $c \in \mathbb{N}, c > 3$, *and* $c \not\equiv 1 \pmod 3$.

(i) *If c is even,* $S = \langle 3, \frac{c+2}{2}, c+2\rangle$ *is the only symmetric numerical semigroup with multiplicity* 3 *and conductor* c.
(ii) *If c is odd,* $S = \langle 3, \frac{c+5}{2}, c+2\rangle$ *is the only pseudo-symmetric numerical semigroup with multiplicity* 3 *and conductor* c.

Proof. (i) It is easy to see that the given numerical semigroup S is symmetric with multiplicity 3 and conductor c. Consider a symmetric numerical semigroup with multiplicity 3, conductor c and Kunz vector (k_1, k_2). Then

$$\begin{aligned} w(1) + w(2) &= (3k_1 + 1) + (3k_2 + 2) = 3\frac{c}{2} + 3 \\ &= (c + 2) + \left(\frac{c}{2} + 1\right) = \mathcal{M} + \frac{c+2}{2} \end{aligned}$$

by Lemma 2.4. It follows that the nonzero element of $\mathrm{Ap}(S, 3)$ that is not the major is $\frac{c+2}{2}$ and $S = \langle 3, \frac{c+2}{2}, c + 2\rangle$. The proof of (ii) is similar and we omit it. □

3.4. *Numerical semigroups with multiplicity* 4

Let S be a numerical semigroup with multiplicity 4 and conductor c. Let $c \equiv \overline{c} \pmod 4$ and thus $c = 4b + \overline{c}$ with $\overline{c} \in \{0, 2, 3\}$ and $b \in \mathbb{N}$. We will consider the three cases $\overline{c} = 0$, $\overline{c} = 2$ and $\overline{c} = 3$ separately.

Proposition 3.6. *Let S be a numerical semigroup with multiplicity* 4, *conductor* $4b, b \in \mathbb{N}$, *and Kunz vector* (k_1, k_2, k_3). *Then the major Kunz coordinate of S is* $k_3 = b$ *and*

$$S = \langle 4, 4k_1 + 1, 4k_2 + 2, 4b + 3\rangle$$

with $(k_1, k_2) \in A_0 \cup B_0$, *where* A_0, B_0 *are sets of ordered pairs of positive integers defined as*

$$A_0 = \left\{(t,u) : \left\lceil \frac{b}{3} \right\rceil \le t \le \left\lceil \frac{b}{2} \right\rceil - 1, b - t \le u \le 2t\right\},$$

$$B_0 = \left\{(t,u) : \left\lceil \frac{b}{2} \right\rceil \le t \le b, b - t \le u \le b\right\} \setminus \{(b,0)\}.$$

Furthermore,

$$N(4, 4b) = \sum_{t=\lceil \frac{b}{3} \rceil}^{\lceil \frac{b}{2} \rceil - 1} (3t - b + 1) + \sum_{t=\lceil \frac{b}{2} \rceil}^{b} (t+1) - 1.$$

Proof. The major Kunz coordinate of S is $k_3 = b$ by (5) and therefore we have $k_1 \le b, k_2 \le b$. Moreover, we have

$$2k_1 - k_2 \ge 0, \quad k_1 + k_2 - b \ge 0$$

by Lemma 2.2. For the Kunz coordinates k_1 and k_2 of S, (k_1, k_2) is a lattice point in the shaded region in Fig. 1; and conversely, if (k_1, k_2) is a lattice point in that region, then (k_1, k_2, b) is the Kunz vector of a numerical semigroup with multiplicity 4 and conductor $4b$. Therefore, there is a one to one correspondence between the set of numerical semigroups with multiplicity 4 and conductor $4b$ and the set of lattice points in the shaded region of Fig. 1. It follows that the number of numerical semigroups with multiplicity 4 and conductor $4b$ is the number of lattice points in the shaded region. It is easily seen that the set of lattice points in the shaded region is the disjoint union of the sets

$$A_0 = \left\{(t,u) : \left\lceil \frac{b}{3} \right\rceil \le t \le \left\lceil \frac{b}{2} \right\rceil - 1, b - t \le u \le 2t\right\}$$

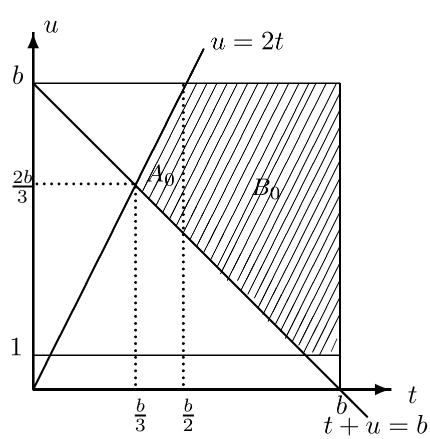

Fig. 1. Region for the Kunz coordinates k_1, k_2.

and

$$B_0 = \left\{(t,u)\colon \left\lceil \frac{b}{2} \right\rceil \le t \le b, b-t \le u \le b\right\} \setminus \{(b,0)\},$$

where t and u denote integers. The descriptions of A_0 and B_0 above can be used to calculate the number of lattice points in these sets. We get

$$N(4,4b) = \sum_{t=\lceil \frac{b}{3} \rceil}^{\lceil \frac{b}{2} \rceil - 1} (3t-b+1) + \sum_{t=\lceil \frac{b}{2} \rceil}^{b} (t+1) - 1. \qquad \square$$

For MED semigroups with multiplicity 4 and conductor $4b$, we have the following proposition.

Proposition 3.7. *Let S be a MED semigroup with multiplicity* 4, *conductor* $4b$, $b \in \mathbb{N}$, *and Kunz vector* (k_1, k_2, k_3). *Then*

$$S = \langle 4, 4k_1+1, 4k_2+2, 4b+3 \rangle$$

with $(k_1,k_2) \in A_0' \cup B_0'$, *where* A_0', B_0' *are sets of ordered pairs of positive integers defined as*

$$A_0' = \left\{(t,u)\colon \left\lceil \frac{b+2}{3} \right\rceil \le t \le \left\lceil \frac{b+1}{2} \right\rceil - 1, b+1-t \le u \le 2t-1\right\},$$

$$B_0' = \left\{(t,u)\colon \left\lceil \frac{b+1}{2} \right\rceil \le t \le b, b+1-t \le u \le b\right\}.$$

Furthermore,

$$N_{\mathrm{MED}}(4,4b) = \sum_{t=\lceil \frac{b+2}{3} \rceil}^{\lceil \frac{b+1}{2} \rceil - 1} (3t-b-1) + \sum_{t=\lceil \frac{b+1}{2} \rceil}^{b} t.$$

Proof. The Kunz coordinates of S being as in Proposition 3.6, S is a MED semigroup if and only if $2k_1 - k_2 \ge 1, k_1 + k_2 - b \ge 1$ by Lemma 2.3. Thus, S is a MED semigroup if and only if the point (k_1,k_2) does not lie on any of the lines $u = 2t$ and $t + u = b$ (see Fig. 1) for the two Kunz coordinates k_1 and k_2 of S. The assertions follow from this observation. $\square$

Symmetric numerical semigroups with multiplicity 4 and conductor $4b$ are characterized as follows.

Proposition 3.8. *Let S be a numerical semigroup with multiplicity* 4 *and conductor* $4b$, $b \in \mathbb{N}$. *Then S is symmetric if and only if*

$$S = \langle 4, 4t+1, 4(b-t)+2, 4b+3 \rangle$$

for some $t \in \mathbb{N}$ *with* $\lceil \frac{b}{3} \rceil \leq t \leq b-1$. *Thus,*

$$N_{\mathrm{SYM}}(4, 4b) = b - \left\lceil \frac{b}{3} \right\rceil.$$

Proof. Necessity is obvious. To prove the sufficiency, let S be a symmetric numerical semigroup with multiplicity 4 and conductor $4b$. Then the major Kunz coordinate of S is $k_3 = b$ and $k_1 \leq b$, $k_2 \leq b$, $2k_1 - k_2 \geq 0$ by Proposition 3.6. Moreover, $k_1 + k_2 + b = \frac{c}{2} = 2b$ by Lemma 2.4. Hence $k_1 + k_2 = b$ and we note that

$$1 \leq k_2 = b - k_1 \Rightarrow k_1 \leq b-1 \quad \text{and} \quad 0 \leq 2k_1 - k_2 = 2k_1 - (b - k_1) \Rightarrow k_1 \geq \frac{b}{3}.$$

Setting $k_1 = t$, the Kunz vector of S is $(t, b-t, b)$ and

$$S = \langle 4, 4t+1, 4(b-t)+2, 4b+3 \rangle, \quad \left\lceil \frac{b}{3} \right\rceil \leq t \leq b-1. \qquad \square$$

Example 3.9. Let us consider numerical semigroups with multiplicity 4 and conductor 16. Then $b = 4$ and Kunz vector of any numerical semigroup with multiplicity 4 and conductor 16 is of the form $(k_1, k_2, 19)$, where k_1 and k_2 satisfy the inequalities

$$1 \leq k_1 \leq 4, \quad 1 \leq k_2 \leq 4, \quad 2k_1 \geq k_2, \quad k_1 + k_2 \geq 4$$

and thus they are the cartesian coordinates of a lattice point of the shaded region in Fig. 2.

There are 11 lattice points in that region. The corresponding Kunz coordinates are

$$\underline{(2,2,4)}, \quad (2,3,4), \quad \underline{(2,4,4)}, \quad \underline{(3,1,4)}, \quad (3,2,4),$$
$$(3,3,4), \quad (3,4,4), \quad (4,1,4), \quad (4,2,4), \quad (4,3,4), \quad (4,4,4),$$

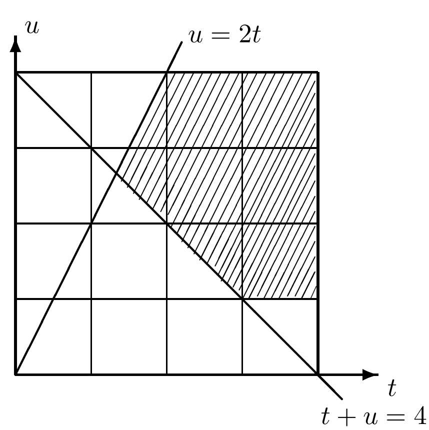

Fig. 2. Lattice points.

and the corresponding numerical semigroups are

$$\underline{\langle 4,9,10,19\rangle}, \quad \langle 4,9,14,19\rangle, \quad \underline{\langle 4,9,18,19\rangle}, \quad \underline{\langle 4,13,6,19\rangle},$$
$$\langle 4,13,10,19\rangle, \quad \langle 4,13,14,19\rangle, \quad \langle 4,13,18,19\rangle, \quad \langle 4,17,6,19\rangle, \quad \langle 4,17,10,19\rangle,$$
$$\langle 4,17,14,19\rangle, \quad \langle 4,17,18,19\rangle.$$

The three underlined semigroups in the above list are not MED semigroups; the remaining eight semigroups in the list are MED semigroups. Two of the semigroups in the list are symmetric: $\langle 4,9,10,19\rangle$ and $\langle 4,13,6,19\rangle$.

Proposition 3.10. *Let S be a numerical semigroup with multiplicity 4, conductor $4b+2$, $b \in \mathbb{N}$, and Kunz vector (k_1,k_2,k_3). Then the major Kunz coordinate of S is $k_1 = b+1$ and*

$$S = \langle 4, 4b+5, 4k_2+2, 4k_3+3\rangle$$

with $(k_2,k_3) \in A_2 \cup B_2$, where A_2, B_2 are sets of ordered pairs of positive integers defined as

$$A_2 = \left\{(t,u)\colon \left\lceil \frac{b-1}{3}\right\rceil \le u \le \left\lceil \frac{b-1}{2}\right\rceil - 1, b-u \le t \le 2u+1\right\},$$
$$B_2 = \left\{(t,u)\colon \left\lceil \frac{b-1}{2}\right\rceil \le u \le b, b-u \le t \le b\right\} \setminus \{(0,b)\}.$$

Furthermore,

$$N(4,4b+2) = \sum_{u=\lceil \frac{b-1}{3}\rceil}^{\lceil \frac{b-1}{2}\rceil - 1} (3u-b+2) + \sum_{u=\lceil \frac{b-1}{2}\rceil}^{b} (u+1) - 1.$$

Proof. The major Kunz coordinate of S is $k_1 = b+1$ by (5) and therefore we have

$$k_2 \le b, \quad k_3 \le b.$$

Moreover, we have

$$2k_3 - k_2 \ge -1, \quad k_2 + k_3 - b \ge 0$$

by Lemma 2.2. The rest of the proof can be given just as in the proof of Proposition 3.6. □

We omit the proofs of the following propositions about MED semigroups and symmetric numerical semigroups with multiplicity 4 and conductor $4b+2$.

Proposition 3.11. *Let S be a MED semigroup with multiplicity 4, conductor $4b+2$, $b \in \mathbb{N}$, and Kunz vector (k_1,k_2,k_3). Then*

$$S = \langle 4, 4b+5, 4k_2+2, 4k_3+3\rangle$$

with $(k_2, k_3) \in A'_2 \cup B'_2$, *where* A'_2, B'_2 *are sets of ordered pairs of positive integers defined as*

$$A'_2 = \left\{(t,u) : \left\lceil \frac{b+1}{3} \right\rceil \le u \le \left\lceil \frac{b}{2} \right\rceil - 1, b+1-u \le t \le 2u \right\},$$

$$B'_2 = \left\{(t,u) : \left\lceil \frac{b}{2} \right\rceil \le u \le b, b+1-u \le t \le b \right\}.$$

Furthermore,

$$N_{\mathrm{MED}}(4, 4b+2) = \sum_{u=\lceil \frac{b+1}{3} \rceil}^{\lceil \frac{b}{2} \rceil - 1} (3u - b) + \sum_{u=\lceil \frac{b}{2} \rceil}^{b} u.$$

Proposition 3.12. *Let* $b \in \mathbb{N}$, $b > 1$. *Then* S *is a symmetric numerical semigroup with multiplicity* 4 *and conductor* $4b+2$ *if and only if*

$$S = \langle 4, 4b+5, 4(b-t)+2, 4t+3 \rangle$$

for some $t \in \mathbb{N}$ *with* $\lceil \frac{b-1}{3} \rceil \le t \le b-1$. *Thus, for any integer* $b > 1$,

$$N_{\mathrm{SYM}}(4, 4b+2) = b - \left\lceil \frac{b-1}{3} \right\rceil.$$

For numerical semigroups with multiplicity 4 and conductor $4b+3$, we have the following results.

Proposition 3.13. *Let* S *be a numerical semigroup with multiplicity* 4, *conductor* $4b+3$, $b \in \mathbb{N}$, *and Kunz vector* (k_1, k_2, k_3). *Then the major Kunz coordinate of* S *is* $k_2 = b+1$ *and*

$$S = \langle 4, 4k_1+1, 4b+6, 4k_3+3 \rangle$$

with

$$(k_1, k_3) \in \left\{(t,u) : \left\lceil \frac{b+1}{2} \right\rceil \le t \le b+1, \left\lceil \frac{b}{2} \right\rceil \le u \le b \right\}.$$

Furthermore,

$$N(4, 4b+3) = \left(b - \left\lceil \frac{b+1}{2} \right\rceil + 2\right)\left(b - \left\lceil \frac{b}{2} \right\rceil + 1\right).$$

Proof. The major Kunz coordinate of S is $k_2 = b+1$ by (5) and therefore we have

$$k_1 \le b+1, \quad k_3 \le b.$$

Moreover, we have

$$2k_1 - (b+1) \ge 0, \quad 2k_3 - b \ge 0$$

by Lemma 2.2. For the Kunz coordinates k_1 and k_3 of S, (k_1, k_3) is a lattice point in the shaded region in Fig. 3; and conversely, if (k_1, k_3) is a lattice point in that region,

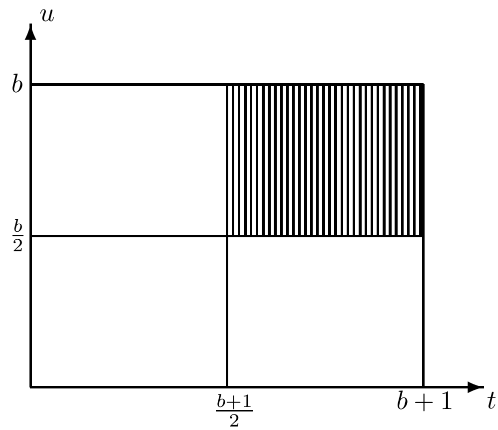

Fig. 3. Region for the Kunz coordinates k_1, k_3.

then the vector $(k_1, b+1, k_3)$ is the Kunz vector of a numerical semigroup with multiplicity 4 and conductor $4b+3$. Therefore, there is a one to one correspondence between the set of numerical semigroups with multiplicity 4 and conductor $4b+3$ and the set of lattice points in the shaded region of Fig. 3. The latter is precisely

$$\left\{(t,u)\colon \left\lceil \frac{b+1}{2} \right\rceil \leq t \leq b+1, \left\lceil \frac{b}{2} \right\rceil \leq u \leq b \right\}$$

and its cardinality gives the formula for $N(4, 4b+3)$:

$$N(4, 4b+3) = \left(b - \left\lceil \frac{b+1}{2} \right\rceil + 2\right)\left(b - \left\lceil \frac{b}{2} \right\rceil + 1\right). \qquad \square$$

MED semigroups and pseudo-symmetric numerical semigroups with multiplicity 4 and conductor $4b+3$ are characterized as follows.

Proposition 3.14. *Let S be a MED semigroup with multiplicity* 4, *conductor* $4b+3$, $b \in \mathbb{N}$, *and Kunz vector* (k_1, k_2, k_3). *Then*

$$S = \langle 4, 4k_1+1, 4b+6, 4k_3+3 \rangle$$

with $(k_1, k_3) \in \{(t,u)\colon \lceil \frac{b+2}{2} \rceil \leq t \leq b+1, \lceil \frac{b+1}{2} \rceil \leq u \leq b\}$. *Thus,*

$$N_{\mathrm{MED}}(4, 4b+3) = \left(b - \left\lceil \frac{b+2}{2} \right\rceil + 2\right)\left(b - \left\lceil \frac{b+1}{2} \right\rceil + 2\right).$$

Proposition 3.15. *For any* $b \in \mathbb{N}$, $S = \langle 4, 4\lceil \frac{b+1}{2} \rceil + 1, 4b+6, 4(b+1-\lceil \frac{b+1}{2} \rceil) + 3 \rangle$ *is the only pseudo-symmetric numerical semigroup with multiplicity* 4 *and conductor* $4b+3$.

Proof. It is clear that the semigroup given in the proposition is pseudo-symmetric with multiplicity 4 and conductor $4b+3$. To prove that it is the only one, let

S be a pseudo-symmetric numerical semigroup with multiplicity 4 and conductor $4b+3$. Then the major Kunz coordinate of S is $k_2 = b+1$ and the remaining Kunz coordinates satisfy $k_1 \geq \frac{b+1}{2}$, $k_3 \geq \frac{b}{2}$ by Proposition 3.13. Moreover, $k_1+b+1+k_3 = \frac{c+1}{2} = 2b+2$ and hence $k_1 + k_3 = b+1$ by Lemma 2.4. The last identity together with $k_3 \geq \frac{b}{2}$ implies $k_1 \leq \frac{b+2}{2}$. Hence $\frac{b+1}{2} \leq k_1 \leq \frac{b+2}{2}$ and therefore $k_1 = \lceil \frac{b+1}{2} \rceil$, $k_3 = b+1-\lceil \frac{b+1}{2} \rceil$. This completes the proof. □

3.5. *Numerical semigroups with multiplicity* 5

Let S be a numerical semigroup with multiplicity 5 and conductor c. Let $c \equiv \overline{c} \pmod 5$ and thus $c = 5b + \overline{c}$ with $\overline{c} \in \{0, 2, 3, 4\}$ and $b \in \mathbb{N}$. We will consider the four cases $\overline{c} = 0$, $\overline{c} = 2$, $\overline{c} = 3$ and $\overline{c} = 4$ separately.

Kunz coordinates other than the major Kunz coordinate of a semigroup with multiplicity 5 constitute the coordinates of lattice points in a polytope in the 3-dimensional Cartesian space. We will sketch that polytope for the case $\overline{c} = 0$ and obtain the parametrization by means of a partition of that polytope. The picture for each of the other cases is very similar to this case.

Proposition 3.16. *Let S be a numerical semigroup with multiplicity* 5, *conductor* $5b, b \in \mathbb{N}$, *and Kunz vector* (k_1, k_2, k_3, k_4). *Then the major Kunz coordinate of S is* $k_4 = b$ *and*

$$S = \langle 5, 5k_1+1, 5k_2+2, 5k_3+3, 5b+4 \rangle$$

with $(k_1, k_2, k_3) \in A_0 \cup B_0 \cup C_0 \cup D_0$, *where* A_0, B_0, C_0, D_0 *are sets of ordered triples of positive integers defined as*

$$A_0 = \left\{ (t,u,v) \colon \left\lceil \frac{2b+1}{3} \right\rceil \leq t \leq b, \left\lceil \frac{b}{2} \right\rceil \leq u \leq b, \left\lceil \frac{t-1}{2} \right\rceil \leq v \leq b \right\},$$

$$B_0 = \left\{ (t,u,v) \colon \left\lceil \frac{b}{2} \right\rceil \leq u \leq \left\lceil \frac{2b}{3} \right\rceil - 1, \left\lceil \frac{u}{2} \right\rceil \leq t \leq b-u, b-t \leq v \leq t+u \right\},$$

$$C_0 = \left\{ (t,u,v) \colon \left\lceil \frac{b}{2} \right\rceil \leq u \leq \left\lceil \frac{2b}{3} \right\rceil - 1, b+1-u \leq t \leq \left\lceil \frac{2b+1}{3} \right\rceil - 1, b-t \leq v \leq b \right\},$$

$$D_0 = \left\{ (t,u,v) \colon \left\lceil \frac{2b}{3} \right\rceil \leq u \leq b, \left\lceil \frac{u}{2} \right\rceil \leq t \leq \left\lceil \frac{2b+1}{3} \right\rceil - 1, b-t \leq v \leq b \right\}.$$

Furthermore,

$$N(5,5b) = \left(b - \left\lceil \frac{b}{2} \right\rceil + 1 \right) \sum_{t=\lceil \frac{2b+1}{3} \rceil}^{b} \left(b - \left\lceil \frac{t-1}{2} \right\rceil + 1 \right) + \sum_{u=\lceil \frac{b}{2} \rceil}^{\lceil \frac{2b}{3} \rceil - 1} \sum_{t=\lceil \frac{u}{2} \rceil}^{b-u} (2t+u-b+1)$$
$$+ \sum_{u=\lceil \frac{b}{2} \rceil}^{\lceil \frac{2b}{3} \rceil - 1} \sum_{t=b+1-u}^{\lceil \frac{2b+1}{3} \rceil - 1} (t+1) + \sum_{u=\lceil \frac{2b}{3} \rceil}^{b} \sum_{t=\lceil \frac{u}{2} \rceil}^{\lceil \frac{2b+1}{3} \rceil - 1} (t+1).$$

Proof. The major Kunz coordinate of S is $k_4 = b$ by (5) and therefore

$$k_1 \leq b, \quad k_2 \leq b, \quad k_3 \leq b.$$

Moreover, we have

$$2k_1 - k_2 \geq 0, \quad k_1 + k_2 - k_3 \geq 0,$$
$$k_1 + k_3 - b \geq 0, \quad 2k_2 - b \geq 0, \quad 2k_3 - k_1 \geq -1$$

by Lemma 2.2. For the Kunz coordinates k_1, k_2 and k_3 of S, (k_1, k_2, k_3) is a lattice point in the polytope in Fig. 4 below; and conversely, if (k_1, k_2, k_3) is a lattice point in that polytope, then (k_1, k_2, k_3, b) is the Kunz vector of a numerical semigroup with multiplicity 5 and conductor $5b$. We get the partition of that polytope as $A_0 \cup B_0 \cup C_0 \cup D_0$ by cutting it with the plane $t = \lceil \frac{2b+1}{3} \rceil$. The portion of the polytope lying in the half-space $t \geq \lceil \frac{2b+1}{3} \rceil$ is denoted by A_0. Description of A_0 as in the proposition is obtained by considering its projection onto the tv-plane. The other portion that lies in the half-space $t \leq \lceil \frac{2b+1}{3} \rceil + 1$ is divided into three mutually disjoint pieces B_0, C_0, D_0 by considering projection onto the tu-plane. The sum of cardinalities of these sets gives the formula for $N(5, 5b)$. □

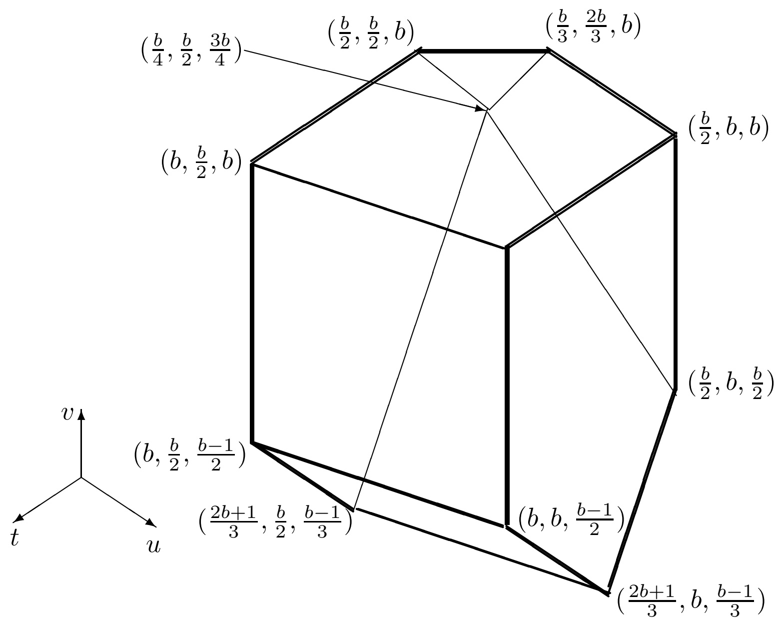

Fig. 4. Polytope for the Kunz coordinates k_1, k_2, k_3.

Using Lemma 2.3 instead of Lemma 2.2 in the above proof, we obtain the following result for MED semigroups with multiplicity 5 and conductor $5b$.

Proposition 3.17. *Let S be a MED semigroup with multiplicity* 5, *conductor* $5b$, $b \in \mathbb{N}$, *and Kunz vector* (k_1, k_2, k_3, k_4). *Then the major Kunz coordinate of S is* $k_4 = b$ *and*

$$S = \langle 5, 5k_1 + 1, 5k_2 + 2, 5k_3 + 3, 5b + 4 \rangle$$

with $(k_1, k_2, k_3) \in A'_0 \cup B'_0 \cup C'_0 \cup D'_0$, *where* A'_0, B'_0, C'_o, D'_0 *are sets of ordered triples of positive integers defined as*

$$A'_0 = \left\{ (t, u, v) \colon \left\lceil \frac{2b+2}{3} \right\rceil \le t \le b, \left\lceil \frac{b+1}{2} \right\rceil \le u \le b, \left\lceil \frac{t}{2} \right\rceil \le v \le b \right\},$$

$$B'_0 = \left\{ (t, u, v) \colon \left\lceil \frac{b+1}{2} \right\rceil \le u \le \left\lceil \frac{2b+1}{3} \right\rceil - 1, \right.$$

$$\left. \left\lceil \frac{u+1}{2} \right\rceil \le t \le b + 1 - u, b + 1 - t \le v \le t + u - 1 \right\},$$

$$C'_0 = \left\{ (t, u, v) \colon \left\lceil \frac{b+1}{2} \right\rceil \le u \le \left\lceil \frac{2b+1}{3} \right\rceil - 1, \right.$$

$$\left. b + 2 - u \le u \le \left\lceil \frac{2b+2}{3} \right\rceil - 1, b + 1 - t \le v \le b \right\},$$

$$D'_0 = \left\{ (t, u, v) \colon \left\lceil \frac{2b+1}{3} \right\rceil \le u \le b, \left\lceil \frac{u+1}{2} \right\rceil \le t \le \left\lceil \frac{2b+2}{3} \right\rceil - 1, \right.$$

$$\left. b + 1 - t \le v \le b \right\}.$$

Furthermore,

$$N_{\mathrm{MED}}(5, 5b) = \left(b - \left\lceil \frac{b+1}{2} \right\rceil + 1 \right) \sum_{t = \lceil \frac{2b+2}{3} \rceil}^{b} \left(b - \left\lceil \frac{t}{2} \right\rceil + 1 \right)$$
$$+ \sum_{u = \lceil \frac{b+1}{2} \rceil}^{\lceil \frac{2b+1}{3} \rceil - 1} \sum_{t = \lceil \frac{u+1}{2} \rceil}^{b+1-u} (2t + u - b - 1)$$
$$+ \sum_{u = \lceil \frac{b+1}{2} \rceil}^{\lceil \frac{2b+1}{3} \rceil - 1} \sum_{t = b+2-u}^{\lceil \frac{2b+2}{3} \rceil - 1} (t) + \sum_{u = \lceil \frac{2b+1}{3} \rceil}^{b} \sum_{t = \lceil \frac{u+1}{2} \rceil}^{\lfloor \frac{2b+2}{3} \rfloor - 1} (t).$$

Proposition 3.18. *Let S be a numerical semigroup with multiplicity* 5, *conductor* $5b, b \in \mathbb{N}$. *Then*

(i) *S is symmetric if and only if b is even and*

$$S = \left\langle 5, 5t + 1, 5\frac{b}{2} + 2, 5(b - t) + 3, 5b + 4 \right\rangle$$

for some $t \in \mathbb{N}$ *with* $\lceil \frac{b}{4} \rceil \le t \le \lfloor \frac{2b+1}{3} \rfloor$. *In that case,*

$$N_{\text{SYM}}(5,5b) = \left\lfloor \frac{2b+1}{3} \right\rfloor - \left\lceil \frac{b}{4} \right\rceil + 1.$$

(ii) S *is pseudo-symmetric if and only if* b *is odd,* $b > 1$, *and*

$$S = \left\langle 5, 5t+1, 5\frac{b+1}{2} + 2, 5(b-t)+3, 5b+4 \right\rangle$$

for some $t \in \mathbb{N}$ *with* $\lceil \frac{b+1}{4} \rceil \le t \le \lfloor \frac{2b+1}{3} \rfloor$. *In that case,*

$$N_{\text{PSYM}}(5,5b) = \left\lfloor \frac{2b+1}{3} \right\rfloor - \left\lceil \frac{b+1}{4} \right\rceil + 1.$$

Proof. It suffices to prove the necessity of both (i) and (ii). Let S be a symmetric numerical semigroup with multiplicity 5 and conductor $5b$. Then the major Kunz coordinate of S is $k_4 = b, 1 \le k_1, k_2, k_3 \le b$ and

$$2k_1 - k_2 \ge 0, \quad k_1 + k_2 - k_3 \ge 0,$$
$$k_1 + k_3 - b \ge 0, \quad 2k_2 - b \ge 0, \quad 2k_3 - k_1 \ge -1$$

by Proposition 3.16 and its proof.

(i) If S is symmetric, then b is even and we have

$$k_1 + k_2 + k_3 + b = \frac{c}{2} = 5\frac{b}{2} \text{ or,} \quad \text{equivalently, } k_1 + k_2 + k_3 = \frac{3b}{2}$$

by Lemma 2.4. Let us note that

$$k_1 + k_3 - b \ge 0, \quad 2k_2 - b \ge 0, \quad k_1 + k_2 + k_3 = \frac{3b}{2} \Rightarrow k_2 = \frac{b}{2}.$$

Thus, $k_1 + k_3 = b$, and

$$k_2 = \frac{b}{2}, \quad 2k_1 - k_2 \ge 0 \Rightarrow k_1 \ge \frac{b}{4},$$
$$1 \le k_3 = b - k_1 \Rightarrow k_1 \le b-1, \quad 2k_3 - k_1 \ge -1 \Rightarrow k_1 \le \frac{2b+1}{3}.$$

We have $\lfloor \frac{2b+1}{3} \rfloor \le b-1$ for $b \ge 2$. Now, setting $k_1 = t$, the Kunz vector of S is $(t, \frac{b}{2}, b-t, b)$ and

$$S = \left\langle 5, 5t+1, 5\frac{b}{2} + 2, 5(b-t)+3, 5b+4 \right\rangle$$

where $\lceil \frac{b}{4} \rceil \le t \le \lfloor \frac{2b+1}{3} \rfloor$. The statement about $N_{\text{SYM}}(5,5b)$ is clear.

(ii) If S is pseudo-symmetric, then b is odd, and $b > 1$ because the numerical semigroup with multiplicity 5 and conductor 5 is not pseudo-symmetric. We have

$$k_1 + k_2 + k_3 + b = \frac{c}{2} = \frac{5b+1}{2} \text{ or,} \quad \text{equivalently, } k_1 + k_2 + k_3 = \frac{3b+1}{2}$$

by Lemma 2.4. Let us note that

$$k_1 + k_3 - b \geq 0, \quad 2k_2 - b \geq 0, \quad k_1 + k_2 + k_3 = \frac{3b+1}{2} \Rightarrow \frac{b}{2} \leq k_2 \leq \frac{b+1}{2}.$$

Since b is an odd integer, this yields $k_2 = \frac{b+1}{2}$ and $k_1 + k_3 = b$. Moreover,

$$k_2 = \frac{b+1}{2}, \quad 2k_1 - k_2 \geq 0 \Rightarrow k_1 \geq \frac{b+1}{4},$$

$$1 \leq k_3 = b - k_1 \Rightarrow k_1 \leq b - 1, \quad 2k_3 - k_1 \geq -1 \Rightarrow k_1 \leq \frac{2b+1}{3}.$$

We have $\lfloor \frac{2b+1}{3} \rfloor \leq b - 1$ for $b > 1$. Now, setting $k_1 = t$, the Kunz vector of S is $(t, \frac{b}{2}, b - t, b)$ and

$$S = \left\langle 5, 5t + 1, 5\frac{b+1}{2} + 2, 5(b-t) + 3, 5b + 4 \right\rangle$$

where $\lceil \frac{b+1}{4} \rceil \leq t \leq \lfloor \frac{2b+1}{3} \rfloor$. The statement about $N_{\mathrm{PSYM}}(5, 5b)$ is clear. □

Example 3.19. Let us consider numerical semigroups with multiplicity 5 and conductor 15. Then $b = 3$ and by Proposition 3.16,

$$S = \langle 5, 5k_1 + 1, 5k_2 + 2, 5k_3 + 3, 19 \rangle$$

with $(k_1, k_2, k_3) \in A_0 \cup D_0$, where A_0, D_0 are sets of ordered triples of positive integers defined as

$$A_0 = \{(3, u, v) : 2 \leq u \leq 3, 1 \leq v \leq 3\},$$
$$D_0 = \left\{(t, u, v) : 2 \leq u \leq 3, \left\lceil \frac{u}{2} \right\rceil \leq t \leq 2, 3 - t \leq v \leq 3\right\}.$$

The sets B_0 and C_0 in Proposition 3.16 are empty. A_0 has 6 elements $(3,2,1), (3,2,2), (3,2,3), (3,3,1), (3,3,2)$ and $(3,3,3)$ yielding the semigroups

$$\langle 5,16,12,8,19 \rangle, \quad \langle \underline{5,16,12,13,19} \rangle, \quad \langle \underline{5,16,12,18,19} \rangle,$$
$$\langle 5,16,17,8,19 \rangle, \quad \langle \underline{5,16,17,13,19} \rangle, \quad \langle \underline{5,16,17,18,19} \rangle;$$

D_0 has 8 elements $(1,2,2)$, $(1,2,3)$, $(2,2,1)$, $(2,2,2)$, $(2,2,3)$, $(2,3,1)$, $(2,3,2)$ and $(2,3,3)$ yielding the following semigroups:

$$\langle 5,6,12,13,19 \rangle^*, \quad \langle 5,6,12,18,19 \rangle, \quad \langle 5,11,12,8,19 \rangle^*, \quad \langle \underline{5,11,12,13,19} \rangle,$$
$$\langle \underline{5,11,12,18,19} \rangle, \quad \langle 5,11,17,8,19 \rangle, \quad \langle \underline{5,11,17,13,19} \rangle, \quad \langle \underline{5,11,17,18,19} \rangle.$$

Thus, there are 14 numerical semigroups with multiplicity 5 and conductor 15. The eight underlined numerical semigroups are MED semigroups, and the two semigroups with * are pseudo-symmetric numerical semigroups with multiplicity 5 and conductor 15. These could also be obtained directly by using Propositions 3.17 and 3.18.

We state our results about the remaining cases without proof.

Proposition 3.20. *Let S be a numerical semigroup with multiplicity* 5, *conductor* $5b+2, b \in \mathbb{N}$, *and Kunz vector* (k_1, k_2, k_3, k_4). *Then the major Kunz coordinate of* S *is* $k_1 = b+1$ *and*

$$S = \langle 5, 5b+6, 5k_2+2, 5k_3+3, 5k_4+4 \rangle$$

with $(k_2, k_3, k_4) \in A_2 \cup B_2 \cup C_2 \cup D_2$, *where* A_2, B_2, C_2, D_2 *are sets of ordered triples of positive integers defined as*

$$A_2 = \left\{ (t,u,v) \colon \left\lceil \frac{b}{2} \right\rceil \leq u \leq b, \left\lceil \frac{2b}{3} \right\rceil \leq v \leq b, \left\lceil \frac{v}{2} \right\rceil \leq t \leq b \right\},$$

$$B_2 = \left\{ (t,u,v) \colon \left\lceil \frac{b}{2} \right\rceil \leq u \leq \left\lceil \frac{2b-1}{3} \right\rceil - 1, \left\lceil \frac{u-1}{2} \right\rceil \leq v \leq b-1-u, \right.$$
$$\left. b-v \leq t \leq u+v+1 \right\},$$

$$C_2 = \left\{ (t,u,v) \colon \left\lceil \frac{b}{2} \right\rceil \leq u \leq \left\lceil \frac{2b-1}{3} \right\rceil - 1, b-u \leq v \leq \left\lceil \frac{2b}{3} \right\rceil - 1, b-v \leq t \leq b \right\},$$

$$D_2 = \left\{ (t,u,v) \colon \left\lceil \frac{2b-1}{3} \right\rceil \leq u \leq b, \left\lceil \frac{u-1}{2} \right\rceil \leq v \leq \left\lceil \frac{2b}{3} \right\rceil - 1, b-v \leq t \leq b \right\}.$$

Furthermore,

$$N(5, 5b+2) = \left(b - \left\lceil \frac{b}{2} \right\rceil + 1 \right) \sum_{v=\lceil \frac{2b}{3} \rceil}^{b} \left(b - \left\lceil \frac{v}{2} \right\rceil + 1 \right)$$
$$+ \sum_{u=\lceil \frac{b}{2} \rceil}^{\lceil \frac{2b-1}{3} \rceil - 1} \sum_{v=\lceil \frac{u-1}{2} \rceil}^{b-1-u} (2v+u-b+2)$$
$$+ \sum_{u=\lceil \frac{b}{2} \rceil}^{\lceil \frac{2b-1}{3} \rceil - 1} \sum_{v=b-u}^{\lceil \frac{2b}{3} \rceil - 1} (v+1) + \sum_{u=\lceil \frac{2b-1}{3} \rceil}^{b} \sum_{v=\lceil \frac{u-1}{2} \rceil}^{\lceil \frac{2b}{3} \rceil - 1} (v+1).$$

Proposition 3.21. *Let S be a MED semigroup with multiplicity* 5, *conductor* $5b+2, b \in \mathbb{N}$, *and Kunz vector* (k_1, k_2, k_3, k_4). *Then the major Kunz coordinate of S is* $k_1 = b+1$ *and*

$$S = \langle 5, 5b+6, 5k_2+2, 5k_3+3, 5k_4+4 \rangle$$

with $(k_2, k_3, k_4) \in A'_2 \cup B'_2 \cup C'_2 \cup D'_2$, *where* A'_2, B'_2, C'_2, D'_2 *are sets of ordered triples of positive integers defined as*

$$A'_2 = \left\{(t,u,v)\colon \left\lceil \frac{b+1}{2} \right\rceil \le u \le b, \left\lceil \frac{2b-1}{3} \right\rceil \le v \le b, \left\lceil \frac{v+1}{2} \right\rceil \le t \le b\right\},$$

$$B'_2 = \left\{(t,u,v)\colon \left\lceil \frac{b+1}{2} \right\rceil \le u \le \left\lceil \frac{2b}{3} \right\rceil - 1, \left\lceil \frac{u}{2} \right\rceil \le v \le b-u,\right.$$
$$\left. b+1-v \le t \le u+v\right\},$$

$$C'_2 = \left\{(t,u,v)\colon \left\lceil \frac{b+1}{2} \right\rceil \le u \le \left\lceil \frac{2b}{3} \right\rceil - 1, b+1-u \le v \le \left\lceil \frac{2b-1}{3} \right\rceil - 1,\right.$$
$$\left. b+1-v \le t \le b\right\},$$

$$D'_2 = \left\{(t,u,v)\colon \left\lceil \frac{2b}{3} \right\rceil \le u \le b, \left\lceil \frac{u}{2} \right\rceil \le v \le \left\lceil \frac{2b-1}{3} \right\rceil - 1, b+1-v \le t \le b\right\}.$$

Furthermore,

$$N_{\mathrm{MED}}(5, 5b+2) = \left(b - \left\lceil \frac{b+1}{2} \right\rceil + 1\right) \sum_{v=\lceil \frac{2b-1}{3} \rceil}^{b} \left(b - \left\lceil \frac{v+1}{2} \right\rceil + 1\right)$$
$$+ \sum_{u=\lceil \frac{b+1}{2} \rceil}^{\lceil \frac{2b}{3} \rceil - 1} \sum_{v=\lceil \frac{u}{2} \rceil}^{b-u} (2v+u-b)$$
$$+ \sum_{u=\lceil \frac{b+1}{2} \rceil}^{\lceil \frac{2b}{3} \rceil - 1} \sum_{v=b+1-u}^{\lceil \frac{2b-1}{3} \rceil - 1} (v) + \sum_{u=\lceil \frac{2b}{3} \rceil}^{b} \sum_{v=\lceil \frac{u}{2} \rceil}^{\lceil \frac{2b-1}{3} \rceil - 1} (v).$$

Proposition 3.22. *Let* S *be a numerical semigroup with multiplicity* 5, *conductor* $5b+2, b \in \mathbb{N}$. *Then*

(i) S *is symmetric if and only if* b *is even and*

$$S = \left\langle 5, 5b+6, 5(b-t)+2, 5\frac{b}{2}+3, 5t+4 \right\rangle$$

for some $t \in \mathbb{N}$ *with* $\lceil \frac{b-2}{4} \rceil \le t \le \lfloor \frac{2b}{3} \rfloor$. *Thus,* $N_{\mathrm{SYM}}(5, 12) = 1$ *and for any even integer* $b > 2$,

$$N_{\mathrm{SYM}}(5, 5b+2) = \left\lfloor \frac{2b}{3} \right\rfloor - \left\lceil \frac{b-2}{4} \right\rceil + 1.$$

(ii) S *is pseudo-symmetric if and only if* b *is odd and*

$$S = \left\langle 5, 5b+6, 5(b-t)+2, 5\frac{b+1}{2}+3, 5t+4 \right\rangle$$

for some $t \in \mathbb{N}$ *with* $\lceil \frac{b-1}{4} \rceil \le t \le \lfloor \frac{2b}{3} \rfloor$. *Thus for any odd integer* $b > 1$,

$$N_{\mathrm{PSYM}}(5, 5b+2) = \left\lfloor \frac{2b}{3} \right\rfloor - \left\lceil \frac{b-1}{4} \right\rceil + 1.$$

Proposition 3.23. *Let* S *be a numerical semigroup with multiplicity* 5, *conductor* $5b+3, b \in \mathbb{N}$, *and Kunz vector* (k_1, k_2, k_3, k_4). *Then the major Kunz coordinate of* S *is* $k_2 = b+1$ *and*

$$S = \langle 5, 5k_1+1, 5b+7, 5k_3+3, 5k_4+4 \rangle$$

with $(k_1, k_3, k_4) \in A_3 \cup B_3 \cup C_3 \cup D_3$, *where* A_3, B_3, C_3, D_3 *are sets of ordered triples of positive integers defined as*

$$A_3 = \left\{ (t,u,v) \colon \left\lceil \frac{b+1}{2} \right\rceil \le t \le b+1, \left\lceil \frac{2b+1}{3} \right\rceil \le u \le b, \left\lceil \frac{u-1}{2} \right\rceil \le v \le b \right\},$$

$$B_3 = \left\{ (t,u,v) \colon \left\lceil \frac{b+1}{2} \right\rceil \le t \le \left\lceil \frac{2b+1}{3} \right\rceil - 1, \left\lceil \frac{t-1}{2} \right\rceil \le u \le b-t, \right.$$
$$\left. b-u \le v \le t+u \right\},$$

$$C_3 = \left\{ (t,u,v) \colon \left\lceil \frac{b+1}{2} \right\rceil \le t \le \left\lceil \frac{2b+1}{3} \right\rceil - 1, b+1-t \le u \le \left\lceil \frac{2b+1}{3} \right\rceil - 1, \right.$$
$$\left. b-u \le v \le b \right\},$$

$$D_3 = \left\{ (t,u,v) \colon \left\lceil \frac{2b+1}{3} \right\rceil \le t \le b+1, \left\lceil \frac{t-1}{2} \right\rceil \le u \le \left\lceil \frac{2b+1}{3} \right\rceil - 1, \right.$$
$$\left. b-u \le v \le b \right\}.$$

Furthermore,

$$N(5, 5b+3) = \left(b - \left\lceil \frac{b+1}{2} \right\rceil + 2 \right) \sum_{u=\lceil \frac{2b+1}{3} \rceil}^{b} \left(b - \left\lceil \frac{u-1}{2} \right\rceil + 1 \right)$$
$$+ \sum_{t=\lceil \frac{b+1}{2} \rceil}^{\lceil \frac{2b+1}{3} \rceil - 1} \sum_{u=\lceil \frac{t-1}{2} \rceil}^{b-t} (2u+t-b+1)$$
$$+ \sum_{t=\lceil \frac{b+1}{2} \rceil}^{\lceil \frac{2b+1}{3} \rceil - 1} \sum_{u=b+1-t}^{\lceil \frac{2b+1}{3} \rceil - 1} (u+1) + \sum_{t=\lceil \frac{2b+1}{3} \rceil}^{b+1} \sum_{u=\lceil \frac{t-1}{2} \rceil}^{\lceil \frac{2b+1}{3} \rceil - 1} (u+1).$$

Proposition 3.24. *Let* S *be a MED semigroup with multiplicity* 5, *conductor* $5b+3, b \in \mathbb{N}$, *and Kunz vector* (k_1, k_2, k_3, k_4). *Then the major Kunz coordinate of* S *is*

$k_2 = b+1$ *and*

$$S = \langle 5, 5k_1+1, 5b+7, 5k_3+3, 5k_4+4\rangle$$

with $(k_1,k_3,k_4) \in A'_3 \cup B'_3 \cup C'_3 \cup D'_3$, *where* A'_3, B'_3, C'_3, D'_3 *are sets of ordered triples of positive integers defined as*

$$A'_3 = \left\{(t,u,v) \colon \left\lceil \frac{b+2}{2} \right\rceil \le t \le b+1, \left\lceil \frac{2b+2}{3} \right\rceil \le u \le b, \left\lceil \frac{u}{2} \right\rceil \le v \le b\right\},$$

$$B'_3 = \left\{(t,u,v) \colon \left\lceil \frac{b+2}{2} \right\rceil \le t \le \left\lceil \frac{2b+2}{3} \right\rceil - 1, \left\lceil \frac{t}{2} \right\rceil \le u \le b+1-t,\right.$$
$$\left. b+1-u \le v \le t+u-1\right\},$$

$$C'_3 = \left\{(t,u,v) \colon \left\lceil \frac{b+2}{2} \right\rceil \le t \le \left\lceil \frac{2b+2}{3} \right\rceil - 1, b+2-t \le u \le \left\lceil \frac{2b+2}{3} \right\rceil - 1,\right.$$
$$\left. b+1-u \le v \le b\right\},$$

$$D'_3 = \left\{(t,u,v) \colon \left\lceil \frac{2b+2}{3} \right\rceil \le t \le b+1, \left\lceil \frac{t}{2} \right\rceil \le u \le \left\lceil \frac{2b+2}{3} \right\rceil - 1,\right.$$
$$\left. b+1-u \le v \le b\right\}.$$

Furthermore,

$$N_{\mathrm{MED}}(5, 5b+3) = \left(b - \left\lceil \frac{b+2}{2} \right\rceil + 2\right) \sum_{u=\lceil \frac{2b+2}{3} \rceil}^{b} \left(b - \left\lceil \frac{u}{2} \right\rceil + 1\right)$$
$$+ \sum_{t=\lceil \frac{b+2}{2} \rceil}^{\lceil \frac{2b+2}{3} \rceil - 1} \sum_{u=\lceil \frac{t}{2} \rceil}^{b+1-t} (2u+t-b-1)$$
$$+ \sum_{t=\lceil \frac{b+2}{2} \rceil}^{\lceil \frac{2b+2}{3} \rceil - 1} \sum_{u=b+2-t}^{\lceil \frac{2b+2}{3} \rceil - 1} (u) + \sum_{t=\lceil \frac{2b+2}{3} \rceil}^{b+1} \sum_{u=\lceil \frac{t}{2} \rceil}^{\lceil \frac{2b+2}{3} \rceil - 1} (u).$$

Proposition 3.25. *Let* S *be a numerical semigroup with multiplicity* 5, *conductor* $5b+3, b \in \mathbb{N}$. *Then*

(i) S *is symmetric if and only if* b *is odd and*

$$S = \left\langle 5, 5\frac{b+1}{2} + 1, 5b+7, 5t+3, 5(b-t)t+4 \right\rangle$$

for some $t \in \mathbb{N}$ *with* $\lceil \frac{b-1}{4} \rceil \le t \le \lfloor \frac{2b+1}{3} \rfloor$. *In that case,*

$$N_{\mathrm{SYM}}(5, 5b+3) = \left\lfloor \frac{2b+1}{3} \right\rfloor - \left\lceil \frac{b-1}{4} \right\rceil + 1.$$

(ii) *S is pseudo-symmetric if and only if b is even and*

$$S = \left\langle 5, 5\frac{b+2}{2} + 1, 5b+7, 5t+3, 5(b-t)t+4 \right\rangle$$

for some $t \in \mathbb{N}$ *with* $\lceil \frac{b}{4} \rceil \le t \le \lfloor \frac{2b+1}{3} \rfloor$. *In that case,*

$$N_{\mathrm{PSYM}}(5, 5b+3) = \left\lfloor \frac{2b+1}{3} \right\rfloor - \left\lceil \frac{b-1}{4} \right\rceil + 1.$$

Proposition 3.26. *Let S be a numerical semigroup with multiplicity* 5, *conductor* $5b+4, b \in \mathbb{N}$, *and Kunz vector* (k_1, k_2, k_3, k_4). *Then the major Kunz coordinate of S is* $k_3 = b+1$ *and*

$$S = \langle 5, 5k_1+1, 5k_2+2, 5b+8, 5k_4+4 \rangle$$

with $(k_1, k_2, k_4) \in A_4 \cup B_4 \cup C_4 \cup D_4$, *where* A_4, B_4, C_4, D_4 *are sets of ordered triples of positive integers defined as small*

$$A_4 = \left\{ (t,u,v) \colon \left\lceil \frac{b}{2} \right\rceil \le v \le b, \left\lceil \frac{2b+2}{3} \right\rceil \le u \le b+1, \left\lceil \frac{u}{2} \right\rceil \le t \le b+1 \right\},$$

$$B_4 = \left\{ (t,u,v) \colon \left\lceil \frac{b}{2} \right\rceil \le v \le \left\lceil \frac{2b}{3} \right\rceil - 1, \left\lceil \frac{v}{2} \right\rceil \le u \le b-v, \right.$$
$$\left. b+1-u \le t \le u+v+1 \right\},$$

$$C_4 = \left\{ (t,u,v) \colon \left\lceil \frac{b}{2} \right\rceil \le v \le \left\lceil \frac{2b}{3} \right\rceil - 1, b+1-v \le u \le \left\lceil \frac{2b+2}{3} \right\rceil - 1, \right.$$
$$\left. b+1-u \le t \le b+1 \right\},$$

$$D_4 = \left\{ (t,u,v) \colon \left\lceil \frac{2b}{3} \right\rceil \le v \le b, \left\lceil \frac{v}{2} \right\rceil \le u \le \left\lceil \frac{2b+2}{3} \right\rceil - 1, b+1-u \le t \le b+1 \right\}.$$

Furthermore,

$$N(5, 5b+4) = \left(b - \left\lceil \frac{b}{2} \right\rceil + 1 \right) \sum_{u=\lceil \frac{2b+2}{3} \rceil}^{b+1} \left(b - \left\lceil \frac{u}{2} \right\rceil + 2 \right)$$
$$+ \sum_{v=\lceil \frac{b}{2} \rceil}^{\lceil \frac{2b}{3} \rceil - 1} \sum_{u=\lceil \frac{v}{2} \rceil}^{b-v} (2u+v-b+1)$$
$$+ \sum_{v=\lceil \frac{b}{2} \rceil}^{\lceil \frac{2b}{3} \rceil - 1} \sum_{u=b+1-v}^{\lceil \frac{2b+2}{3} \rceil - 1} (u) + \sum_{v=\lceil \frac{2b}{3} \rceil}^{b} \sum_{u=\lceil \frac{v}{2} \rceil}^{\lceil \frac{2b+2}{3} \rceil - 1} (u).$$

Proposition 3.27. *Let S be a MED semigroup with multiplicity* 5, *conductor* $5b+4, b \in \mathbb{N}$, *and Kunz vector* (k_1, k_2, k_3, k_4). *Then the major Kunz coordinate of S is*

$k_3 = b+1$ *and*

$$S = \langle 5, 5k_1+1, 5k_2+2, 5b+8, 5k_4+4\rangle$$

with $(k_1,k_2,k_4) \in A'_4 \cup B'_4 \cup C'_4 \cup D'_4$, *where* A'_4, B'_4, C'_4, D'_4 *are sets of ordered triples of positive integers defined as*

$$A'_4 = \left\{(t,u,v)\colon \left\lceil \frac{b+1}{2}\right\rceil \le v \le b, \left\lceil \frac{2b+3}{3}\right\rceil \le u \le b+1, \left\lceil \frac{u+1}{2}\right\rceil \le t \le b+1\right\},$$

$$B'_4 = \left\{(t,u,v)\colon \left\lceil \frac{b+1}{2}\right\rceil \le v \le \left\lceil \frac{2b+1}{3}\right\rceil - 1, \left\lceil \frac{v+1}{2}\right\rceil \le u \le b+1-v, \right.$$
$$\left. b+2-u \le t \le u+v\right\},$$

$$C'_4 = \left\{(t,u,v)\colon \left\lceil \frac{b+1}{2}\right\rceil \le v \le \left\lceil \frac{2b+1}{3}\right\rceil - 1, b+2-v \le u \le \left\lceil \frac{2b+3}{3}\right\rceil - 1, \right.$$
$$\left. b+2-u \le t \le b+1\right\},$$

$$D'_4 = \left\{(t,u,v)\colon \left\lceil \frac{2b+1}{3}\right\rceil \le v \le b, \left\lceil \frac{v+1}{2}\right\rceil \le u \le \left\lceil \frac{2b+3}{3}\right\rceil - 1, \right.$$
$$\left. b+2-u \le t \le b+1\right\}.$$

Furthermore,

$$N_{\mathrm{MED}}(5,5b+4) = \left(b - \left\lceil \frac{b+1}{2}\right\rceil + 1\right) \sum_{u=\lceil \frac{2b+3}{3}\rceil}^{b+1} \left(b - \left\lceil \frac{u+1}{2}\right\rceil + 2\right)$$
$$+ \sum_{v=\lceil \frac{b+1}{2}\rceil}^{\lceil \frac{2b+1}{3}\rceil - 1} \sum_{u=\lceil \frac{v+1}{2}\rceil}^{b+1-v} (2u+v-b-1)$$
$$+ \sum_{v=\lceil \frac{b+1}{2}\rceil}^{\lceil \frac{2b+1}{3}\rceil - 1} \sum_{u=b+2-v}^{\lceil \frac{2b+3}{3}\rceil - 1} (u) + \sum_{v=\lceil \frac{2b+1}{3}\rceil}^{b} \sum_{u=\lceil \frac{v+1}{2}\rceil}^{\lceil \frac{2b+3}{3}\rceil - 1} (u).$$

Proposition 3.28. *Let S be a numerical semigroup with multiplicity* 5, *conductor* $5b+4, b \in \mathbb{N}$. *Then*

(i) *S is symmetric if and only if b is even and*

$$S = \left\langle 5, 5(b+1-t)+1, 5t+2, 5b+8, 5\frac{b}{2}+4\right\rangle$$

for some $t \in \mathbb{N}$ with $\lceil \frac{b}{4}\rceil \le t \le \lfloor \frac{2b+2}{3}\rfloor$. In that case,

$$N_{\mathrm{SYM}}(5,5b+4) = \left\lfloor \frac{2b+2}{3}\right\rfloor - \left\lceil \frac{b}{4}\right\rceil + 1.$$

(ii) *S is pseudo-symmetric if and only if b is odd and*

$$S = \left\langle 5, 5(b+1-t)+1, 5t+2, 5b+8, 5\frac{b+1}{2}+4 \right\rangle$$

for some $t \in \mathbb{N}$ *with* $\lceil \frac{b+1}{4} \rceil \leq t \leq \lfloor \frac{2b+2}{3} \rfloor$. *In that case,*

$$N_{\mathrm{PSYM}}(5, 5b+4) = \left\lfloor \frac{2b+2}{3} \right\rfloor - \left\lceil \frac{b+1}{4} \right\rceil + 1.$$

The following example appears in [14].

Example 3.29. Let us consider numerical semigroups with multiplicity 5 and conductor 14. Then $b = 2$ and by Proposition 3.26,

$$S = \langle 5, 5k_1+1, 5k_2+2, 18, 5k_4+4 \rangle$$

with $(k_1, k_2, k_4) \in A_4 \cup B_4 \cup D_4$, where A_4, B_4, D_4 are sets of ordered triples of positive integers defined as

$$\begin{aligned}
A_4 &= \left\{(t,u,v) : 1 \leq v \leq 2, 2 \leq u \leq 3, \left\lceil \frac{u}{2} \right\rceil \leq t \leq 3\right\}, \\
B_4 &= \left\{(t,u,v) : 1 \leq v \leq 1, \left\lceil \frac{v}{2} \right\rceil \leq u \leq 2-v, 3-u \leq t \leq u+v+1\right\}, \\
C_4 &= \{(t,u,v) : 1 \leq v \leq 1, 3-v \leq u \leq 1, 3-u \leq t \leq 3\}, \\
D_4 &= \left\{(t,u,v) : 2 \leq v \leq 2, \left\lceil \frac{v}{2} \right\rceil \leq u \leq 1, 3-u \leq t \leq 3\right\}.
\end{aligned}$$

The set C_4 in Proposition 3.26 is empty. A_4 has 10 elements $(1,2,1), (2,2,1)$, $(3,2,1)$, $(2,3,1)$, $(3,3,1)$, $(1,2,2)$, $(2,2,2)$, $(3,2,2)$, $(2,3,2)$, $(3,3,2)$ yielding the semigroups $\langle 5,6,12,18,9\rangle^*$, $\langle 5,11,12,18,9\rangle$, $\langle 5,16,12,18,9\rangle$, $\langle 5,11,17,18,9\rangle$, $\langle 5,16,17,18,9\rangle$, $\langle 5,6,12,18,14\rangle$, $\underline{\langle 5,11,12,18,14\rangle}$, $\underline{\langle 5,16,12,18,14\rangle}$, $\underline{\langle 5,11,17,18,14\rangle}$, $\underline{\langle 5,16,17,18,14\rangle}$; B_4 has two elements $(2,1,1), (3,1,1)$ yielding the two semigroups $\langle 5,11,7,18,9\rangle^*$, $\langle 5,16,7,18,9\rangle$; D_4 has two elements $(2,1,2)$ and $(3,1,2)$ yielding the two semigroups $\langle 5,11,7,18,14\rangle$ and $\langle 5,16,7,18,14\rangle$. Thus, there are 14 numerical semigroups with multiplicity 5 and conductor 14. The four underlined numerical semigroups are MED semigroups, and the two semigroups with * are symmetric numerical semigroups with multiplicity 5 and conductor 14. These could also be obtained directly by using Propositions 3.27 and 3.28.

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