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Coefficient estimates for the class of quasi q-convex functions

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Abstract: In this paper we introduce and investigate the class of $P_q(\lambda, \beta, A, B)$, which is called quasi q-starlike and quasi q-convex with respect to the values of the parameter λ. We give coefficient bounds estimates and the results for the main theorem.

Key words: q-Derivative, starlike function, convex function, q-starlike function, q-convex function, quasi q-starlike function, quasi q-convex function, close-to-quasi q-starlike function, close-to-quasi q-convex function

1. Introduction

Let A denote the class of analytic functions in the open unit disc $U = \{z \in C : |z| < 1\}$ of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, a_n \geq 0.$$

We say that the function $f(z)$ is subordinate to $g(z)$ and can be represented as $f \prec g$, if there exists a function $w(z)$ such that $w(0) = 0$, $|w(z)| < 1$, and $f(z) = g(w(z))$. If $g(z)$ is univalent the above subordination is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$ (see [3]). $S^*(\alpha)$ and $C(\alpha)$ are starlike and convex functions of order α respectively such that

$$S^*(\alpha) = \left\{ f \in A : Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, z \in U \right\}$$

$$C(\alpha) = \left\{ f \in A : Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, z \in U \right\}$$

for $\alpha = 0$, $S^* = S^*(0)$ and $C = C(0)$ are respectively starlike and convex functions in U (see [3]). Jackson (see [4]) introduced q-derivative operator of the functions $f(z)$ as follows:

$$D_q f(z) = \begin{cases} \frac{f(z)-f(qz)}{(1-q)z}, & if \quad z \neq 0 \\ f'(0), \quad if & z = 0 \end{cases} \tag{1.1}$$

for $q \in (0, 1)$. It is clear that

$$lim_{q \to 1^-} D_q f(z) = f'(z).$$

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For $n \in N$, $D_q z^n = [n]_q z^{n-1}$, $[n]_q = \frac{1-q^n}{1-q}$, $[0]_q = 0$. Recently Uçar Özkan (see [8]) studied some properties of q-starlike and q-close to convex functions. Ramachandran [6] studied q-starlike and q-convex functions with respect to symmetric points. Polatoğlu [5] investigated $q-$starlike functions and obtained growth and distortion theorems for this class.
Quasi-starlike and quasi-convex functions were studied by Altıntaş (see [1]). Altıntaş and Mustafa studied q-starlike and q-convex functions (see [2]).
We have the following properties:

Remark 1.1 *We have symbols for the following classes:*

i. $CV(\beta)$ *is the class of convex functions of order* β.

ii. $ST(\beta)$ *is the class of starlike functions of order* β.

iii. $CCV(\alpha)$ *is the class of close-to-convex of order* α.

iv. $CST(\alpha)$ *is the class of close-to-starlike of order* α.

v. $QCV(\alpha)$ *is the class of quasi-convex of order* α.

vi. $QST(\alpha)$ *is the class of quasi-starlike of order* α.

vii. $CC_qV(\alpha)$ *is the class of close-to-*q *convex of order* α.

viii. $CS_qT(\alpha)$ *is the class of close-to-*q *starlike order* α.

ix. $QC_qV(\alpha)$ *is the class of quasi-*q *convex of order* α.

x. $QS_qT(\alpha)$ *is the class of quasi-*q *starlike of order* α.

Remark 1.2 *We have definitions of the following classes.*

i. $CV = \left\{ f \in A : Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \right\}$.

ii. $ST = \left\{ f \in A : Re\frac{zf'(z)}{f(z)} > 0 \right\}$.

iii. $CV(\beta) = \left\{ f \in A : Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta, \quad 0 \leq \beta < 1 \right\}$.

iv. $ST(\beta) = \left\{ f \in A : Re\frac{zf'(z)}{f(z)} > \beta, \quad 0 \leq \beta < 1 \right\}$.

v. $CCV(\beta) = \left\{ f \in A : Re\frac{f'(z)}{g'(z)} > \beta, \quad 0 \leq \beta < 1, \quad g \in CV \right\}$.

vi. $CST(\beta) = \left\{ f \in A : Re\frac{f(z)}{g(z)} > \beta, \quad 0 \leq \beta < 1, \quad g \in ST \right\}$.

vii. $CCV(\beta, A, B) = \left\{ f \in A : \frac{f'(z)}{g'(z)} \prec \frac{1+Az}{1+Bz}, \quad g \in CV(\beta), \quad -1 \leq B < A \leq 1 \right\}$.

viii. $CST(\beta, A, B) = \left\{ f \in A : \frac{f(z)}{g(z)} \prec \frac{1+Az}{1+Bz}, \quad g \in ST(\beta), \quad -1 \leq B < A \leq 1 \right\}$.

ix. $QCV(\beta, A, B) = \left\{ f \in A : \frac{(zf'(z))'}{g'(z)} \prec \frac{1+Az}{1+Bz}, \quad g \in CV(\beta), \quad -1 \leq B < A \leq 1 \right\}$.

x. $QST(\beta, A, B) = \left\{ f \in A : \frac{zf'(z)}{g(z)} \prec \frac{1+Az}{1+Bz}, \quad g \in ST(\beta), \quad -1 \leq B < A \leq 1 \right\}$.

xi. $QC_qV(\beta, A, B) = \left\{ f \in A : \frac{D_q(zD_q(z))}{D_qg(z)} \prec \frac{1+Az}{1+Bz}, \quad g \in CV(\beta), \quad -1 \leq B < A \leq 1, \quad q \in (0,1) \right\}$.

xii. $QS_qT(\beta, A, B) = \left\{ f \in A : \frac{zD_qf(z)}{g(z)} \prec \frac{1+Az}{1+Bz}, \quad g \in ST(\beta), \quad -1 \leq B < A \leq 1, \quad q \in (0,1) \right\}$.

Remark 1.3 *i.* $f \in CV(\beta) \Rightarrow zf' \in ST(\beta)$, $(0 \leq \beta < 1)$.

ii. $f \in CCV(\alpha) \Rightarrow zf' \in CST(\alpha)$, $(0 \leq \alpha < 1)$.

iii. $f \in QCV(\alpha) \Rightarrow zf' \in QST(\alpha)$.

iv. $f \in QC_qV(\alpha) \Rightarrow zf' \in QS_qT(\alpha)$.

Lemma 1.4 *If $h(z) = 1 + c_1z + c_2z^2 + ...$ is analytic in U and $h(z) \prec \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$, then we have*

$$Reh(z) > \frac{1-A}{1-B}. \tag{1.2}$$

(see [7]).

Lemma 1.5 *If $f(z) \in C_qV(\beta)$ or $Re\left[1 + \frac{zD_q^2f(z)}{D_qf(z)}\right] > \beta$, $(0 \leq \beta < 1)$, then*

$$\sum_{n=2}^{\infty}([n]_q - \beta)[n]_q a_n < 1 - \beta.$$

For the proof of this lemma, we use the following equation:

$$D_q^2 f(z) = D_q(D_qf(z)). \tag{1.3}$$

(see[2].Corallary 2.5)

Definition 1.6 *A function $f(z) \in A$ in the form of (1.1) is said to be in the class $P_q(\lambda, \beta, A, B)$ if the following condition is satisfied:*

$$\frac{D_qf(z) + \lambda z D_q^2 f(z)}{D_qg(z)} \prec \frac{1+Az}{1+Bz}, \tag{1.4}$$

where $g(z) \in CV(\beta)$, $q \in (0,1)$, $0 \leq \lambda \leq 1$, $0 \leq \beta < 1$, $-1 \leq B < A \leq 1$.
If $\lambda = 0$ then we have

$$\frac{D_qf(z)}{D_qg(z)} = \frac{zD_qf(z)}{zD_qg(z)} = \frac{zD_qf(z)}{h(z)} \prec \frac{1+Az}{1+Bz},$$

and $h(z) \in ST(\beta)$, *so* $f(z) \in QS_qT(\beta, A, B)$.
If $\lambda = 1$ *then we have*

$$\frac{D_q(zD_qf(z))}{D_qg(z)} \prec \frac{1+Az}{1+Bz},$$

and $g(z) \in CV(\beta)$. *Hence,* $f(z) \in QC_qV(\beta, A, B)$.

2. Main results

Theorem 2.1 *If* $f(z) \in P_q(\lambda, \beta, A, B)$ *then we have*

$$\sum_{n=2}^{\infty}[n]_q[1+\lambda[n-1]_q]a_n \leq 1-\alpha+\alpha\frac{1-\beta}{[2]_q-\beta}, \tag{2.1}$$

where $q \in (0,1)$, $0 \leq \lambda \leq 1$, $0 \leq \beta < 1$, $\alpha = \frac{1-A}{1-B}$, $-1 \leq B < A \leq 1$.

Proof From Lemma 1.3 and Definition 1.5 we have

$$h(z) \prec \frac{1+Az}{1+Bz} \Rightarrow Reh(z) > \frac{1-A}{1-B} = \alpha$$

and

$$Re\frac{1-\sum_{n=2}^{\infty}[n]_qa_nz^{n-1}-\lambda\sum_{n=2}^{\infty}[n]_q[n-1]_qa_nz^{n-1}}{1-\sum_{n=2}^{\infty}[n]_qb_nz^{n-1}} > \alpha.$$

If we choose z real and $z \to 1^-$ then we have

$$\sum_{n=2}^{\infty}[n]_q[1+\lambda[n-1]_q]a_n \leq 1-\alpha+\alpha\sum_{n=2}^{\infty}[n]_qb_n. \tag{2.2}$$

From Lemma 1.4 if $g(z) \in C_qV(\beta)$ and $g(z) = z - \sum_{n=2}^{\infty}b_nz^n$, $(b_n \geq 0)$, then we let

$$\sum_{n=2}^{\infty}([n]_q-\beta)[n]_qb_n \leq 1-\beta,$$

or

$$([2]_q-\beta)\sum_{n=2}^{\infty}[n]_qb_n \leq 1-\beta,$$

and

$$\sum_{n=2}^{\infty}[n]_qb_n \leq \frac{1-\beta}{[2]_q-\beta}. \tag{2.3}$$

Using (2.3) in (2.2) we find the equality (2.1). The result is sharp for the function

$$f(z) = f_n(z) = z - \frac{1-\alpha+\alpha\frac{1-\beta}{[2]_q-\beta}}{[n]_q[1+\lambda][n-1]_q}z^n.$$

□

From Theorem 2.1 for the different values of q, λ, β, A, B we can obtain the following results and all results are sharp.

Corollary 2.2 *If* $f \in QC_qV(\beta, A, B)$ *then we have*

$$\sum_{n=2}^{\infty}[n]_q(1+[n-1]_q)a_n \leq 1-\alpha+\alpha\frac{1-\beta}{[2]_q-\beta}. \tag{2.4}$$

We let $\lambda = 1$ *in Theorem* 2.1*.*

Corollary 2.3 *If* $f \in QCV(\beta, A, B)$ *then we have*

$$\sum_{n=2}^{\infty} n^2 a_n \leq 1-\alpha+\alpha\frac{1-\beta}{2-\beta}, \tag{2.5}$$

We let $q \to 1^-$ *and* $\lambda = 1$ *in Theorem* 2.1*.*

Corollary 2.4 *If* $f \in QCV(0, A, B)$ *then we have*

$$\sum_{n=2}^{\infty} n^2 a_n \leq 1-\alpha+\frac{\alpha}{2}=\frac{2-\propto}{2}. \tag{2.6}$$

We let $q \to 1^-$, $\lambda = 1$ *and* $\beta = 0$ *in Theorem* 2.1*.*

Corollary 2.5 *If* $f \in QCV(0, 1, -1)$ *then we have*

$$\sum_{n=2}^{\infty} n^2 a_n \leq 1 \tag{2.7}$$

We let $q \to 1^-$, $\lambda = 1$, $B = -1$, *and* $A = 1$ *in Theorem* 2.1*.*

Corollary 2.6 *If* $f \in QS_qT(\beta, A, B)$ *then we have*

$$\sum_{n=2}^{\infty}[n]_q a_n \leq 1-\alpha+\alpha\frac{1-\beta}{[2]_q-\beta}. \tag{2.8}$$

We let $\lambda = 0$ *in Theorem* 2.1*.*

Corollary 2.7 *If* $f \in QST(\beta, A, B)$ *then we have*

$$\sum_{n=2}^{\infty} n a_n \leq 1-\alpha+\alpha\frac{1-\beta}{[2]_q-\beta}, \tag{2.9}$$

We let $q \to 1^-$ *and* $\lambda = 0$ *in Theorem* 2.1*.*

Corollary 2.8 *If $f \in QST(0, A, B)$ then we have*

$$\sum_{n=2}^{\infty} na_n \leq (1-\alpha+\frac{\alpha}{2}) = \frac{2-\propto}{2}. \tag{2.10}$$

We let $q \to 1^-$, $\lambda = 0$, and $\beta = 0$ in Theorem 2.1.

Corollary 2.9 *If $f \in QST(0, 1, -1)$ then we have*

$$\sum_{n=2}^{\infty} na_n \leq 1. \tag{2.11}$$

We let $q \to 1^-$, $\lambda = 0$, $\beta = 0$, $A = 1$, and $B = -1$ in Theorem 2.1.

Corollary 2.10 *If $f \in CC_qV(\beta, A, B)$ then we have*

$$\sum_{n=2}^{\infty} [n]_q a_n \leq 1-\alpha+\alpha\frac{1-\beta}{[2]_q-\beta}. \tag{2.12}$$

We let $\lambda = 0$ in Theorem 2.1.

Corollary 2.11 *If $f \in CC_qV(\beta, A, B) \Leftrightarrow f \in QS_qT(\beta, A, B)$ we let $\lambda = 0$ in Theorem 2.1 and then we have*

$$\frac{D_q f(z)}{D_q g(z)} = \frac{zD_q f(z)}{zD_q g(z)} \prec \frac{1+Az}{1+Bz}, \tag{2.13}$$

and $g(z) \in CV(\beta) \Rightarrow zD_q g(z) = h(z)$, or

$$\frac{zD_q f(z)}{h(z)} \prec \frac{1+Az}{1+Bz}, \tag{2.14}$$

and $h(z) \in ST(\beta)$ so $f(z) \in QS_qT(\beta, A, B)$. Hence, we have $CC_qV(\beta, A, B) = QS_qT(\beta, A, B)$.

References

[1] Altıntaş O, Kılıc ÖÖ. Coefficient estimates for a class containing quasi-convex functions. Turkish Journal of Mathematics 2019; 42: 2819-2825.

[2] Altıntaş O, Nizami M. Coefficient bounds and distortion theorems for the certain analytic functions. Turkish Journal of Mathematics 2019; 43: 985-997.

[3] Goodman AW. Univalent Functions, Vol. I and Vol. II. Tampa, FL, USA; Mariner Publishing, 1983.

[4] Jackson FH. On $q-$derivative integrals. Quarterly Journal of Pure and Applied Mathematics 1910; 41: 193-203.

[5] Polatoğlu Y. Growth and distortion theorem for generalized q-starlike functions. Advances in Mathematics Scientific Journal 2016; 5 (1): 7-12.

[6] Ramachandran C, Kavitha D, Soupramanien T. Certain bound for q-starlike and q-convex functions with respect to symmetric points. International Journal of Mathematics and Mathematical Sciences 2015; 2015: 205682.

[7] Srivastava HM, Altıntaş O, Serenbay SK. Coefficient bounds for certain subclasses of starlike functions of complex order. Applied Mathematics Letters 2011; 24 (8): 1359-1363.

[8] Uçar Özkan HE. Coefficient inequality for q-starlike functions. Applied Mathematics and Computation 2016; 276: 122-126.