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The generalized Baskakov type operators

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ABSTRACT

The use of Baskakov type operators is difficult for numerical calculation because these operators include infinite series. Do the operators expressed as a finite sum provide the approximation properties? Furthermore, are they appropriate for numerical calculation? In this paper, in connection with these questions, we define a new family of linear positive operators including finite sum by using the Baskakov type operators. We also give some numerical results in order to compare Baskakov type operators with this new defined operator.

1. Introduction and preliminaries

A general construction of Baskakov operators based on a sequence of functions $\{\varphi_n\}$ $(n = 1, 2, \ldots)$ $\varphi_n : C \to C$, having the following properties

(i) For every $n = 1, 2, \ldots$ φ_n is analytic on a domain D_n, containing the disc $B_n = \{z \in C : |z - b_n| \leq b_n\}$, $\lim_{n\to\infty} b_n = \infty$;

(ii) $\varphi_n(0) = 1$ $(n = 1, 2, \ldots)$;

(iii) φ_n $(n = 1, 2, \ldots)$ is completely monotone on $[0, b_n]$, i.e., $(-1)^k \varphi_n^{(k)}(x) \geq 0$ for any $k = 0, 1, 2, \ldots$;

(iv) there exists a positive integer $m(n)$, such that

$$\varphi_n^{(k)}(x) = -n\varphi_{m(n)}^{(k-1)}(x) \left(1 + \alpha_{k,n}(x)\right), \quad x \in [0, b_n] \quad (n, k = 1, 2, \ldots)$$

where $\alpha_{k,n}(0)$ converges to zero for $n \to \infty$ uniformly in k;

(v) $\lim_{n\to\infty} \frac{n}{m(n)} = 1$.

Under these conditions we will consider the following Baskakov type operators

$$L_n(f, x) = \sum_{k=0}^{\infty} \frac{(-x)^k}{k!} \varphi_n^{(k)}(x) f\left(\frac{k}{n}\right), \quad 0 \leq x \leq b_n. \tag{1}$$

It is obvious that L_n translated a continuous function with the growth condition $f(x) =: O\left(x^2\right)$ at infinity to such a type of function, which may be seen from the properties (2)–(4).

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Lemma 1. *The following equalities hold:*

$$L_n(1,x) = 1, \tag{2}$$

$$L_n(t,x) = \left(1+\alpha_{1,n}(0)\right)x, \tag{3}$$

$$L_n\left(t^2,x\right) = \frac{m(n)}{n}\left(1+\alpha_{1,m(n)}(0)\right)(1+\alpha_{2,n}(0))x^2 + \frac{(1+\alpha_{2,n}(0))}{n}x. \tag{4}$$

Proof. Since φ_n $(n=1,2,\ldots)$ be analytic on a domain D_n, we have

$$\varphi_n(y) = \sum_{k=0}^{\infty}\frac{(y-x)^k}{k!}\varphi_n^{(k)}(x).$$

By condition (ii), for $y=0$ we get

$$L_n(1,x) = \sum_{k=0}^{\infty}\frac{(-x)^k}{k!}\varphi_n^{(k)}(x) = 1.$$

Now, we consider the case $L_n(t,x)$ as follows:

$$\begin{aligned} L_n(t,x) &= \sum_{k=0}^{\infty}\frac{(-x)^k}{k!}\varphi_n^{(k)}(x)\frac{k}{n} \\ &= \frac{-x}{n}\sum_{k=1}^{\infty}\frac{(-x)^{k-1}}{(k-1)!}\varphi_n^{(k)}(x) \\ &= \frac{-x}{n}\sum_{k=0}^{\infty}\frac{(-x)^k}{k!}\varphi_n^{(k+1)}(x). \end{aligned}$$

From the equality $\varphi_n^{(r)}(0) = \sum_{k=0}^{\infty}\frac{(-x)^k}{k!}\varphi_n^{(k+r)}(x)$ we have

$$\varphi_n'(0) = \sum_{k=0}^{\infty}\frac{(-x)^k}{k!}\varphi_n^{(k+1)}(x)$$

therefore we get,

$$L_n(t,x) = \frac{-x}{n}\varphi_n'(0).$$

From condition (iv), we obtained the desired result. (4) can be proved similarly. ■

When $b_n = b$ in (1), in the case when all functions φ_n, $n=1,2,\ldots$, are analytic on the fixed disc $B = \{z \in C : |z-b| \le b\} \subset D$ where D is a domain, the sequence of operators (1) were investigated by many authors (see, for example [1–5]).

But all of these investigations are devoted to the problem of approximation of a function belonging to the class mentioned above and we do not know of any further result on approximation theorems in polynomial weighted spaces, given in [6] for a special Baskakov operator, which may be obtained from (1) in the case of

$$\varphi_n(x) = \frac{1}{(1+x)^n}, \quad x \ge 0,\ n = 1,2,\ldots.$$

In [7], Gadziev and Atakut investigated the approximation of continuous functions having polynomial growth at infinity, by the operator given in (1). They also gave an estimate for a difference $|L_n(f,x) - f(x)|$ on any finite interval through the modulus of continuity of a function f and the theorem on weighted approximation on all positive semi-axes.

Note that a weighted Korovkin's type theorem was proven in [8,9] and we need a special case of this theorem.

Let $B_{2m}[0,\infty)$ be the space of all functions, satisfying the inequality

$$|f(x)| \le M_f\left(1+x^{2m}\right), \quad x \ge 0,\ m \in \mathbb{N} \tag{5}$$

where M_f is constant, depending on a function f and let $C_{2m}[0,\infty)$ consist of all continuous functions belonging to $B_{2m}[0,\infty)$. Let also $C_{2m}^0[0,\infty)$ be a subset of functions in $C_{2m}[0,\infty)$ for which

$$\lim_{x\to\infty}\frac{f(x)}{1+x^{2m}} \tag{6}$$

exists finitely.

Theorem 2. *Let T_n be the sequence of linear positive operators, acting from $C_{2m}[0,\infty)$ to $B_{2m}[0,\infty)$ which satisfy the conditions*

$$\lim_{n\to\infty} \|T_n(t^\upsilon, x) - x^\upsilon\|_{C_{2m}[0,\infty)} = 0, \quad \upsilon = 0, m, 2m$$

where

$$\|f\|_{C_{2m}[0,\infty)} = \sup_{x\geq 0} \frac{|f(x)|}{1+x^{2m}}.$$

Then for any function $f \in C_{2m}^0[0,\infty)$,

$$\lim_{n\to\infty} \|T_n f - f\|_{C_{2m}[0,\infty)} = 0$$

and there exists a function $f^* \in C_{2m}[0,\infty)/C_{2m}^0[0,\infty)$ such that

$$\lim_{n\to\infty} \left\|T_n f^* - f^*\right\|_{C_{2m}[0,\infty)} > 1.$$

Proof. The proof of this theorem can be seen in [8,9]. ■

Lemma 3 ([7]). *For any natural number υ,*

$$L_n(t^\upsilon, x) = \alpha(\upsilon, n)\, x^\upsilon + \sum_{k=1}^{\upsilon-1} \frac{\psi_{k,\upsilon}(x)}{n^k}$$

where $\psi_{k,\upsilon}(x)$ $k = 1, \ldots, \upsilon - 1$ are bounded functions on any finite closed interval and

$$\lim_{n\to\infty} \alpha(\upsilon, n) = 1.$$

Theorem 4 ([7]). *For any function $f \in C_{2m}^0[0,\infty)$,*

$$\lim_{n\to\infty} \sup_{0\leq x\leq b_n} \frac{|L_n(f;x) - f(x)|}{1+x^{2m}} = 0.$$

Lemma 5 ([7]). *For any natural number m,*

$$\lim_{n\to\infty} L_n\left(|t-x|^{2m}; x\right) = 0.$$

2. Construction of the new type L_n operators including finite sum

In this study, inspired by [10], we replace the infinite sum in the generalized Baskakov type operators by a truncated sum. It shows the same approximation properties of L_n operators.

Now we give the generalization of L_n operators of one variable including finite sum based on the above idea.

Definition 6. For $f \in C_{2m}[0,\infty)$, we define the sequence of operators G_n by the formula

$$G_n(f, s_n, x) := \sum_{k=0}^{[n(x+s_n)]} \frac{(-x)^k}{k!} \varphi_n^{(k)}(x) f\left(\frac{k}{n}\right) \tag{7}$$

$$x \in [0,\infty), \quad n \in \mathbb{N}$$

where $(s_n)_1^\infty$ is a sequence of positive numbers such that $\lim_{n\to\infty} s_n = \infty$ and $[n(x+s_n)]$ denotes the integral part of $n(x+s_n)$.

We can easily see that the G_n operator defined by (7) is a sequence of linear positive operators acting from $C_{2m}[0,\infty)$ to $B_{2m}[0,\infty)$.

Now we give the approximation theorem for G_n operators.

Theorem 7. *For $m \in \mathbb{N}$ and G_n defined by (7), we have*

$$\lim_{n\to\infty} G_n(f, s_n, x) = f(x), \quad f \in B_{2m}$$

uniformly on every interval $[x_1, x_2]$, $x_2 > x_1 \geq 0$.

Proof. Let $f \in B_{2m}$ and $m \in \mathbb{N}$. From (1) and (7) we obtain

$$\begin{aligned} G_n(f, s_n, x) - f(x) &= \sum_{k=0}^{[n(x+s_n)]} \frac{(-x)^k}{k!} \varphi_n^{(k)}(x) f\left(\frac{k}{n}\right) - f(x) \\ &= \sum_{k=0}^{\infty} \frac{(-x)^k}{k!} \varphi_n^{(k)}(x) f\left(\frac{k}{n}\right) - f(x) - \sum_{k=[n(x+s_n)]+1}^{\infty} \frac{(-x)^k}{k!} \varphi_n^{(k)}(x) f\left(\frac{k}{n}\right) \\ &= L_n(f, x) - f(x) - R_n(f, s_n, x); \quad x \in R_0, \ n \in \mathbb{N} \end{aligned}$$

where

$$R_n(f, s_n, x) = \sum_{k=[n(x+s_n)]+1}^{\infty} \frac{(-x)^k}{k!} \varphi_n^{(k)}(x) f\left(\frac{k}{n}\right).$$

Using the elementary inequality $(a+b)^k \le 2^{k-1}\left(a^k + b^k\right)$, $a, b > 0$, $k \in \mathbb{N}_0$, we have

$$\begin{aligned} |f(t)| &\le K_1\left(1 + t^{2m}\right) \\ &\le K_1\left(1 + (|t-x| + x)^{2m}\right) \\ &\le K_1\left(1 + 2^{2m-1}\left(|t-x|^{2m} + x^{2m}\right)\right). \end{aligned}$$

Using this inequality and (1), we get

$$\begin{aligned} |R_n(f, s_n, x)| &\le \sum_{k=[n(x+s_n)]+1}^{\infty} \frac{(-x)^k}{k!} \varphi_n^{(k)}(x) \left| f\left(\frac{k}{n}\right)\right| \\ &\le \sum_{k=[n(x+s_n)]+1}^{\infty} \frac{(-x)^k}{k!} \varphi_n^{(k)}(x) K_1 \left(1 + 2^{2m-1}\left(\left|\frac{k}{n} - x\right|^{2m} + x^{2m}\right)\right) \\ &\le K_1\left(\left(1 + 2^{2m-1}x^{2m}\right) \sum_{k=[n(x+s_n)]+1}^{\infty} \frac{(-x)^k}{k!} \varphi_n^{(k)}(x) + 2^{2m-1} \sum_{k=0}^{\infty} \frac{(-x)^k}{k!} \varphi_n^{(k)}(x) \left|\frac{k}{n} - x\right|^{2m}\right) \\ &= K_1\left(\left(1 + 2^{2m-1}x^{2m}\right) \sum_{k=[n(x+s_n)]+1}^{\infty} \frac{(-x)^k}{k!} \varphi_n^{(k)}(x) + |2^{2m-1} L_n\left(|t-x|^{2m}, x\right)\right). \end{aligned}$$

We remark that

$$\begin{aligned} \sum_{k=[n(x+s_n)]+1}^{\infty} \frac{(-x)^k}{k!} \varphi_n^{(k)}(x) &\le \sum_{s_n < |(k/n) - x|}^{\infty} \frac{(-x)^k}{k!} \varphi_n^{(k)}(x) \\ &\le \sum_{s_n < |(k/n) - x|}^{\infty} \frac{(-x)^k}{k!} \varphi_n^{(k)}(x) \frac{|(k/n) - x|^{2m}}{s_n^{2m}} \\ &\le \frac{1}{s_n^{2m}} \sum_{k=0}^{\infty} \frac{(-x)^k}{k!} \varphi_n^{(k)}(x) \left|\frac{k}{n} - x\right|^{2m} \\ &= \frac{1}{s_n^{2m}} L_n\left(|t-x|^{2m}, x\right) \end{aligned}$$

this implies that

$$|R_n(f, s_n, x)| \le K_1\left(\frac{1 + 2^{2m-1}x^{2m}}{s_n^{2m}} + 2^{2m-1}\right) L_n\left(|t-x|^{2m}, x\right).$$

From the $\lim_{n\to\infty} s_n = \infty$ and Lemma 5,

$$\lim_{n\to\infty} R_n(f, s_n, x) = 0$$

uniformly on every interval $[x_1, x_2]$, $x_2 > x_1 \ge 0$, which completes the proof of theorem.

3. Construction of the new $L_{m,n}$ operators of two variables including finite sum

Now, we introduce certain linear positive operators in polynomial weighted spaces of the function of two variables. For natural numbers p, q, let $B_{2p,2q}$ be the space of all functions satisfying the inequality

$$|f(x,y)| \le M_f \left(1+x^{2p}\right)\left(1+y^{2q}\right), \quad x \ge 0,\ y \ge 0$$

where M_f is constant, depending on a function f. We denote by $C_{2p\ ,2q}$ the space of all continuous functions belonging to $B_{2p,2q}$ and by $C^0_{2p\ ,2q}$ a subset of functions in $C_{2p\ ,2q}$ for which

$$\lim_{x,y\to\infty} \frac{f(x,y)}{\left(1+x^{2p}\right)\left(1+y^{2q}\right)}$$

exists finitely.

We define the generalized Baskakov type operators of two variables as the following formula

$$L_{m,n}(f;x,y) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-x)^j}{j!}\frac{(-y)^k}{k!}\varphi_m^{(j)}(x)\varphi_n^{(k)}(y) f\left(\frac{j}{m},\frac{k}{n}\right). \tag{8}$$

The following theorem can be proved, as in the proof of Theorem 4 [7, p. 37].

Theorem 8. *For any function* $f \in C^0_{2p\ ,2q}$

$$\lim_{m,n\to\infty} \sup_{\substack{0\le x\le b_n\\ 0\le y\le b_m}} \frac{\left|L_{m,n}(f;x,y)-f(x,y)\right|}{\left(1+x^{2p}\right)\left(1+y^{2q}\right)} = 0.$$

From this limit value it was deduced that

$$\lim_{m,n\to\infty} L_{m,n}(f;x,y) = f(x,y), \quad (x,y)\in[0,\infty)\times[0,\infty)$$

uniformly on every rectangle $0\le x\le x_0,\ 0\le y\le y_0$.

Now, we give in the following a new type of operators by $L_{m,n}$.

Definition 9. For fixed $p,q\in\mathbb{N}$, we define the sequence of operators $G_{m,n}$ by the formula

$$G_{m,n}(f;s_m,t_n;x,y) := \sum_{j=0}^{[m(x+s_m)]}\sum_{k=0}^{[n(y+t_n)]} \frac{(-x)^j}{j!}\frac{(-y)^k}{k!}\varphi_m^{(j)}(x)\varphi_n^{(k)}(y) f\left(\frac{j}{m},\frac{k}{n}\right)$$

$$f\in C_{2p\ ,2q}, \quad (x,y)\in[0,\infty)\times[0,\infty) \tag{9}$$

where $(s_m)_1^\infty$ and $(t_n)_1^\infty$ are given sequences of positive numbers such that $\lim_{m\to\infty} s_m=\infty$ and $\lim_{n\to\infty} t_n=\infty$.

We can easily see that $G_{m,n}$ operators defined by (9) are a sequence of linear positive operators.

By using Theorem 8 and (8), we can prove the basic property of $G_{m,n}$.

Theorem 10. *For* $n,m\in\mathbb{N}$ *and* $G_{m,n}$ *defined by* (9)*, we have*

$$\lim_{m,n\to\infty} G_{m,n}(f;s_m,t_n;x,y) = f(x,y), \quad f\in C_{2p\ ,2q}$$

uniformly on every rectangle $0\le x\le x_0,\ 0\le y\le y_0$.

Proof. Firstly, we suppose that $f\in C_{2p\ ,2q}$. For $p,q\in\mathbb{N}$, we have

$$\begin{aligned}|f(t,z)| &\le K_2\left(1+t^{2p}\right)\left(1+z^{2q}\right)\\ &\le K_2\left(1+(|t-x|+x)^{2p}\right)\left(1+(|z-y|+y)^{2q}\right)\\ &\le K_2\left(1+2^{2p-1}\left(|t-x|^{2p}+x^{2p}\right)\right)\left(1+2^{2q-1}\left(|z-y|^{2q}+y^{2q}\right)\right).\end{aligned}$$

Using this inequality and (8), we get

$$G_{m,n}(f;s_m,t_n;x,y)-f(x,y) = L_{m,n}(f;x,y)-f(x,y)-R_{m,n}(f;s_m,t_n;x,y)$$

where

$$R_{m,n}(f;s_m,t_n;x,y) = \sum_{j=[m(x+s_m)]+1}^{\infty}\sum_{k=[n(y+t_n)]+1}^{\infty} \frac{(-x)^j}{j!}\varphi_m^{(j)}(x)\frac{(-y)^k}{k!}\varphi_n^{(k)}(y) f\left(\frac{j}{m},\frac{k}{n}\right), \quad (x,y)\in[0,\infty)\times[0,\infty).$$

Table A1

n	15	50	100
$f(x) = e^{-x}$	0.006734699	0.0067346999	0.0067946999
$G_n(f, s_n, x)$	0.010702892	0.0070859750	0.0067944861
$L_n(f, x)$	0.010702903	0.0070859752	0.0067944865

Table A2

m, n	3, 3	7, 7	10, 10
$f(x, y)$	0.2549945975	0.2549945975	0.2549945975
$G_{m,n}(f; s_m, t_n; x, y)$	0.2486221351	0.2582780079	0.2570681365
$L_{m,n}(f; x, y)$	0.2702365567	0.2582874809	0.2570681388

We note that

$$\begin{aligned}\left|R_{m,n}(f; s_m, t_n; x, y)\right| &\le \sum_{j=[m(x+s_m)]+1}^{\infty} \sum_{k=[n(y+t_n)]+1}^{\infty} \frac{(-x)^j}{j!}\varphi_m^{(j)}(x)\frac{(-y)^k}{k!}\varphi_n^{(k)}(y)\left|f\left(\frac{j}{m}, \frac{k}{n}\right)\right| \\ &\le K_2 \sum_{j=[m(x+s_m)]+1}^{\infty} \frac{(-x)^j}{j!}\varphi_m^{(j)}(x)\left(1 + 2^{2p-1}\left(\left|\frac{j}{m} - x\right|^{2p} + x^{2p}\right)\right) \\ &\quad\times \sum_{k=[n(y+t_n)]+1}^{\infty} \frac{(-y)^k}{k!}\varphi_n^{(k)}(y)\left(1 + 2^{2q-1}\left(\left|\frac{k}{n} - y\right|^{2q} + y^{2q}\right)\right).\end{aligned}$$

Performing the same calculations as in the second part of Theorem 7, we obtain

$$\sum_{j=[m(x+s_m)]+1}^{\infty} \frac{(-x)^j}{j!}\varphi_m^{(j)}(x)\left(1 + 2^{2p-1}\left(\left|\frac{j}{m} - x\right|^{2p} + x^{2p}\right)\right) \le \left(\frac{1 + 2^{2p-1}x^{2p}}{s_m^{2p}} + 2^{2p-1}\right) L_{m,n}\left(|t - x|^{2p}, x\right)$$

and

$$\sum_{k=[n(y+t_n)]+1}^{\infty} \frac{(-y)^k}{k!}\varphi_n^{(k)}(y)\left(1 + 2^{2q-1}\left(\left|\frac{k}{n} - y\right|^{2q} + y^{2q}\right)\right) \le \left(\frac{1 + 2^{2q-1}y^{2q}}{t_n^{2q}} + 2^{2q-1}\right) L_{m,n}\left(|z - y|^{2q}, y\right).$$

Using these inequalities, we get

$$\lim_{m,n\to\infty} R_{m,n}(f; s_m, t_n; x, y) = 0$$

uniformly on every rectangle $0 \le x \le x_0,\ 0 \le y \le y_0$.

Remark 11. If we select $\varphi_n(x) = \frac{1}{(1+x)^n}$ and $\varphi_m(y) = \frac{1}{(1+y)^m}$ in Theorem 10, we obtain the results given in [10].

For a given function f, we give two examples to show the values of G_n and $G_{m,n}$ operators including finite sum and the generalized Baskakov operators at x-fixed point.

We also give an example in which we see that the values of the generalized Baskakov operators L_n including infinite sum are not calculated but the values of the operators G_n defined by (7) are calculated at some x-fixed point.

3.1. Some examples

Example 12. For $n = 5, 50, 100$ and $f(x) = e^{-x}$, $x = 5$ and $s_n = n$, some numerical values of $G_n(f, s_n, x)$, $L_n(f, x)$ and $f(x)$ are given together in Table A1.

Example 13. For $n, m = 1, 5, 10$ and $f(x, y) = e^{-\sqrt{x}}\ln(y + 1)$, $x = 1$, $y = 1$ and $s_m = \ln(m)$, $t_n = \ln(n)$, some numerical values of $G_{m,n}(f; s_m, t_n; x, y)$, $L_{m,n}(f; x, y)$ and $f(x, y)$ are given together in Table A2.

Baskakov operators and their various modifications require estimations of infinite series which in a certain sense restrict their usefulness from the computational. Let us show this with a following example.

Example 14. For $n = 10, 50, 100$ and $f(x) = e^{-\sqrt{x}}$, $x = 2$ and $s_n = \ln(n)$, some numerical values of $G_n(f, s_n, x)$ and $f(x)$ are given in Table A3.

Table A3

n	10	50	100
$f(x) = e^{-\sqrt{x}}$	0.24367345	0.24367345	0.24367345
$G_n(f, s_n, x)$	0.24856630	0.24414914	0.24373019
$L_n(f, x)$	Not calculated	Not calculated	Not calculated

Remark 15. Due to the intensive development of q-calculus and its applications in various fields such as mathematics, mechanics, and physics, the applications of q-calculus in the area of approximation theory have emerged. In recent years, the convergence of the q-generalization of linear positive operators and their generalization has been investigated by many authors. See [11] for more details on this topic. The q analogue of the operators G_n and $G_{m,n}$ defined by (7) and (9), respectively, in this paper, can be defined and approximation properties can be studied in an elaborate manner in future studies.

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