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ON THE SECOND ORDER INVOLUTE CURVES IN $\mathbb{E}^3$

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Abstract. In this study we worked on the involute of involute curve of curve α. We called them the second order involute of curve α in $\mathbb{E}^3$. All Frenet apparatus of the second order involute of curve α are examined in terms of Frenet apparatus of the curve α. Further we show that; Frenet vector fields of the second order involute curve α_2 can be written based on the principal normal vector field of curve α. Besides, we illustrate examples of our results.

The involute of the curve is well known by the mathematicians especially the differential geometry scientists. There are many important consequences and properties of curves. Involute curves have been studied by some authors [1, 2, 3, 5]. Let $\alpha : I \to \mathbb{E}^3$ be the C^2- class differentiable unit speed curve denote by $\{T, N, B\}$ the moving Frenet frame. For an arbitrary curve $\alpha \in \mathbb{E}^3$, with first and second curvature, κ and τ respectively, the Frenet formulae are given by [3]

$$\begin{cases} T' = \kappa N \\ N' = -\kappa T + \tau B \\ B' = -\tau N. \end{cases} \tag{0.1}$$

The tangent lines to a curve α generate a surface called the tangent surface of α. A curve α_1 which lies on the tangent surface of α and intersects the tangent lines orthogonally is called an involute of α. The equation of the involutes is,

$$\alpha_1(s) = \alpha(s) + \lambda(s)T(s), \quad \lambda(s) = c - s, \ \ c \in \mathbb{R}, \tag{0.2}$$

where c is constant, [3]. The relationship are between Frenet apparatus of this curves as follows, [5].

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$$\begin{cases} T_1 = N \\ N_1 = \dfrac{-\kappa}{\sqrt{\kappa^2+\tau^2}}T + \dfrac{\tau}{\sqrt{\kappa^2+\tau^2}}B \\ B_1 = \dfrac{\tau}{\sqrt{\kappa^2+\tau^2}}T + \dfrac{\kappa}{\sqrt{\kappa^2+\tau^2}}B, \end{cases} \tag{0.3}$$

and

$$\kappa_1 = \frac{\sqrt{\kappa^2+\tau^2}}{(c-s)\kappa}, \quad \tau_1 = \frac{-\tau^2\left(\frac{\kappa}{\tau}\right)'}{(c-s)\kappa\left(\kappa^2+\tau^2\right)}. \tag{0.4}$$

For any unit speed curve $\alpha : I \to \mathbb{E}^3$, the vector W is called Darboux vector which is defined by [2]

$$W = \tau T + \kappa B. \tag{0.5}$$

If we consider the normalization of the Darboux $C = \frac{1}{\|W\|}W$, we have Figure 1

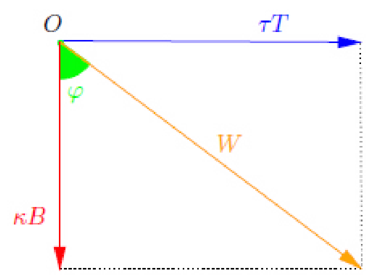

FIGURE 1. Darboux vector

$$\sin\varphi = \frac{\tau}{\sqrt{\kappa^2+\tau^2}} = \frac{\tau}{\|W\|}, \quad \cos\varphi = \frac{\kappa}{\sqrt{\kappa^2+\tau^2}} = \frac{\kappa}{\|W\|} \tag{0.6}$$

and

$$C = \sin\varphi T + \cos\varphi B \tag{0.7}$$

where $\angle(W, B) = \varphi$, [4]. Substituting the equation (0.6) into equation (0.3) and (0.4), we can write [1],

$$\begin{cases} T_1 = N \\ N_1 = -\cos\varphi T + \sin\varphi B \\ B_1 = \sin\varphi T + \cos\varphi B, \end{cases} \tag{0.8}$$

and

$$\kappa_1 = \frac{\sec\varphi}{\lambda}, \qquad \tau_1 = \frac{\varphi'}{\lambda\kappa}. \tag{0.9}$$

1. Second Order Involute Curve

$\alpha_1 : I \to \mathbb{E}^3$ and $\alpha_2 : I \to \mathbb{E}^3$ are the arclengthed curves with the arcparameters s_1 and s_2, respectively. The quantities $\{T_1, N_1, B_1, \kappa_1, \tau_1\}$ and $\{T_2, N_2, B_2, \kappa_2, \tau_2\}$ are collectively Frenet-Serret apparatus of the curve α_1 and the involute α_2, respectively. α_1 has the parametrization with arclength s as the involute curve of $\alpha(s)$. Also α_2 has the parametrization with arclength s as the involute curve of $\alpha_1(s)$, hence we can give the following definitions in terms of the parameter s. Let $\alpha_2(s_2)$ be the involute of the curve $\alpha_1(s)$ then we have the following equation

$$\alpha_2(s) = \alpha_1(s) + \lambda_1 T_1(s). \tag{1.1}$$

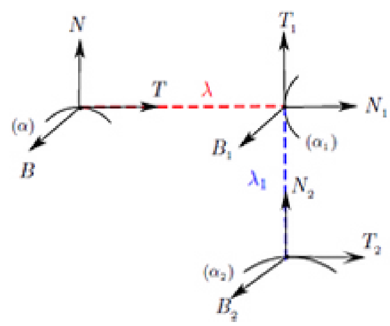

Figure 2. Involute of involute of the curve α

Theorem 1. *The distance between corresponding points of the involute curve α_1 and its involute α_2 curve is*

$$\lambda_1 = c_1 - \int \lambda\kappa ds, \ \ c_1 = constant, \forall s \in I. \tag{1.2}$$

Proof. Differentiating (1.1), we can write

$$\Rightarrow T_2 \frac{ds_2}{ds} = -\lambda_1\kappa T + \left(\lambda_1{}' + \lambda\kappa\right)N + \lambda_1\tau B$$

where $T_1 = N$ and $\langle T_1, T_2\rangle = 0$ is. If we multiply internal both sides of the equation with T_1 we have,

$$\begin{aligned} & \lambda_1{}' + \lambda\kappa = 0 \\ \Rightarrow \quad & \lambda_1{}' = -\lambda\kappa \\ \Rightarrow \quad & \lambda_1 = c_1 - \int \lambda\kappa ds \end{aligned}$$

where $c_1 \in \mathbb{R}$ and c_1 is constant. □

Substituting the equation (0.2) and (0.3) into equation (1.1), this give as following definition:

Definition 1. *$\alpha : I \to \mathbb{E}^3$ be an unit speed curve. If α_1 is an involute of α and α_2 is an involute of α_1, then the curve α_2 is called second order involute curve of α.*

$$\alpha_2(s) = \alpha(s) + \lambda(s)T(s) + \lambda_1(s)N(s) \tag{1.3}$$

is the expression of the second order involute curve α.

Theorem 2. *The Frenet vector fields of the second order involute α_2, based in the Frenet apparatus of the curve α are*

$$\begin{cases} T_2 = \dfrac{-\kappa}{\|W\|}T + \dfrac{\tau}{\|W\|}B \\ N_2 = \dfrac{-1}{\|W\|\sqrt{\|W\|^6 + (\tau^2 n)^2}}\left(\tau^3 nT + \|W\|^4 N + \kappa\tau^2 nB\right) \\ B_2 = \dfrac{1}{\sqrt{\|W\|^6 + (\tau^2 n)^2}}\left(\|W\|^2\tau T - \tau^2 nN + \|W\|^2\kappa B\right) \end{cases} \tag{1.4}$$

Proof. It is easy to say that Frenet vectors of the second order involute α_2, based on the Frenet apparatus of the curve α_1 are

$$\begin{cases} T_2 = N_1 \\ N_2 = \dfrac{-\kappa_1}{\sqrt{{\kappa_1}^2 + {\tau_1}^2}}T_1 + \dfrac{\tau_1}{\sqrt{\kappa_1^2 + \tau_1^2}}B_1 \\ B_2 = \dfrac{\tau_1}{\sqrt{{\kappa_1}^2 + \tau_1^2}}T_1 + \dfrac{\kappa_1}{\sqrt{{\kappa_1}^2 + \tau_1^2}}B_1. \end{cases} \tag{1.5}$$

Substituting (0.3) and (0.4) into equation (1.5), we have

$$T_2 = N_1 = \frac{-\kappa T + \tau B}{\sqrt{\kappa^2 + \tau^2}} = \frac{-\kappa T + \tau B}{\|W\|},$$

$$N_2 = \frac{-\kappa_1 T_1 + \tau_1 B_1}{\sqrt{\kappa_1^2 + \tau_1^2}} = \frac{-1}{\|W\|\sqrt{\|W\|^6 + (\tau^2 n)^2}}\left(\tau^3 nT + \|W\|^4 N + \kappa\tau^2 nB\right)$$

and

$$B_2 = \frac{\tau_1 T_1 + \kappa_1 B_1}{\sqrt{\kappa_1^2 + \tau_1^2}} = \frac{1}{\sqrt{\|W\|^6 + (\tau^2 n)^2}}\left(\|W\|^2\tau T - \tau^2 nN + \|W\|^2\kappa B\right).$$

where $\|W\| = \sqrt{\kappa^2+\tau^2}$ and $\left(\frac{\kappa}{\tau}\right)' = n \neq 0$, which has the following matrix form

$$\begin{bmatrix} T_2 \\ N_2 \\ B_2 \end{bmatrix} = \frac{1}{\|W\|}\begin{bmatrix} -\kappa & 0 & \tau \\ -\frac{\tau^3 n}{\sqrt{\|W\|^6+(\tau^2 n)^2}} & -\frac{\|W\|^4}{\sqrt{\|W\|^6+(\tau^2 n)^2}} & -\frac{\kappa\tau^2 n}{\sqrt{\|W\|^6+(\tau^2 n)^2}} \\ \frac{\|W\|^2\tau}{\sqrt{\|W\|^6+(\tau^2 n)^2}} & -\frac{\tau^2 n}{\sqrt{\|W\|^6+(\tau^2 n)^2}} & \frac{\|W\|^2\kappa}{\sqrt{\|W\|^6+(\tau^2 n)^2}} \end{bmatrix}\begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{1.6}$$

□

Theorem 3. *The first and the second curvatures of the second order involute α_2 based on the Frenet apparatus of the curve α are respectively.*

$$\kappa_2 = \sqrt{\frac{\|W\|^6+(\tau^2 n)^2}{\lambda_1^2\|W\|^6}}, \quad \tau_2 = -\frac{\tau^4 n^2\left(\frac{\tau^2 n}{\|W\|^3}\right)'}{\lambda_1\|W\|\left(\|W\|^6+\tau^4 n^2\right)} \tag{1.7}$$

Proof. In order to calculate the curvature and torsion of the curve α_2, we differentiate

$$\begin{cases} {\alpha_2}' = -\kappa\lambda_1 T + \tau\lambda_1 B, \\ {\alpha_2}'' = \big(-\kappa'\lambda_1 - \lambda\kappa^2\big)T - \|W\|^2\lambda_1 N + \big(\tau'\lambda_1 + \lambda\kappa\tau\big)B, \\ {\alpha_2}''' = \big(-\kappa''\lambda_1 - \kappa\kappa'^2 - \kappa\|W\|^2\lambda_1\big)T - \big(\lambda\kappa^3 + \|W\|^2\lambda\kappa + \kappa\lambda\tau^2\big)N \\ \qquad +\big(\tau''\lambda_1 + \|W\|^2\lambda_1\tau + \lambda\kappa\tau' - \kappa\tau\big)B. \end{cases} \tag{1.8}$$

The curvature of second order involute α_2 is

$$\kappa_2 = \frac{\|{\alpha_2}' \wedge {\alpha_2}''\|}{\|{\alpha_2}'^3},$$

$$\kappa_2 = \sqrt{\frac{\|W\|^6+(\tau^2 n)^2}{\lambda_1^2\|W\|^6}}$$

Also it is easy to say that, the torsion of second order involute α_2 is

$$\begin{aligned} \tau_2 &= \frac{\det\{{\alpha_2}', {\alpha_2}'', {\alpha_2}'''\}}{\|{\alpha_2}' \wedge {\alpha_2}''^2}, \\ \tau_2 &= -\frac{\tau^4 n^2\left(\frac{\tau^2 n}{\|W\|^3}\right)'}{\lambda_1\|W\|\left(\|W\|^6+\tau^4 n^2\right)}. \end{aligned}$$

□

Theorem 4. *Let unit Darboux vector field of involute α_1 be C_1. This vector is expressed in terms of Frenet apparatus of the curve α*

$$C_1 = \frac{1}{\sqrt{\varphi'^2 + (\kappa \sec\varphi)^2}}\Big(\kappa \tan\varphi T + \varphi' N + B\Big) \tag{1.9}$$

Proof. The vector C_1 is the direction of the Darboux vector W_1 of the involute curve α_1 we can write

$$C_1 = \sin\varphi_1 T_1 + \cos\varphi_1 B_1, \tag{1.10}$$

where

$$\cos\varphi_1 = \frac{\kappa_1}{\sqrt{\kappa_1^2 + \tau_1^2}}, \quad \sin\varphi_1 = \frac{\tau_1}{\sqrt{\kappa_1^2 + \tau_1^2}}. \tag{1.11}$$

Substituting the equation (0.9) into equation (1.11), we can write

$$\cos\varphi_1 = \frac{\varphi'}{\sqrt{\varphi'^2 + (\kappa \sec\varphi)^2}}, \quad \sin\varphi_1 = \frac{\kappa \sec\varphi}{\sqrt{\varphi'^2 + (\kappa \sec\varphi)^2}}. \tag{1.12}$$

Substituting the equation (1.12) and (0.8) into equation (1.10), proof is complete. □

Theorem 5. *Let unit Darboux vector field of second order involute curve α_2 be C_2. This vector is expressed in terms of Frenet apparatus curve α*

$$C_2 = \frac{\delta}{\sqrt{1+\eta^2}}\Big((-\delta\cos\varphi + \sin\varphi)T + \frac{\kappa \sec\varphi^2}{\varphi'|c-s|}N + (-\delta\sin\varphi + \cos\varphi)B\Big), \tag{1.13}$$

where

$$\delta = \Big(\frac{\varphi'}{\sqrt{\varphi'^2 + \|W\|^2}}\Big)' \frac{\sqrt{\varphi'^2 + \|W\|^2}}{\|W\|} \ \textit{and}\ \eta = \Big(\frac{\varphi'}{\sqrt{\varphi'^2 + \|W\|^2}}\Big)' \cos\varphi(c-s).$$

Proof. The vector C_2 is the direction of the Darboux vector W_2 of the second order involute curve α_2 Hence we have

$$C_2 = \frac{\tau_2}{\sqrt{\kappa_2^2 + \tau_2^2}}T_2 + \frac{\kappa_2}{\sqrt{\kappa_2^2 + \tau_2^2}}B_2, \tag{1.14}$$

Substituting the equation (1.4) and (1.7) into equation (1.14), we can write

$$C_2 = \frac{\delta}{\sqrt{1+\eta^2}}\Big((-\delta\cos\varphi + \sin\varphi)T + \frac{\kappa \sec\varphi^2}{\varphi'|c-s|}N + (-\delta\sin\varphi + \cos\varphi)B\Big),$$

□

is complete proof.

Corollary 1. *The Frenet vector fields of the involute curve α_1, can be written as the principal normal vector field on the curve α*

$$T_1 = N,\ \ N_1 = \frac{N'}{\|N'\|},\ \ B_1 = T_1 \wedge N_1. \tag{1.15}$$

Corollary 2. *The Frenet vectors of the second order involute α_2 are expressed based on the Frenet apparatus of the curve α are*

$$\begin{cases} T_2 = -\cos\varphi T + \sin\varphi B \\ N_2 = \dfrac{\varphi' \sin\varphi}{\sqrt{\varphi'^2 + (\kappa\sec\varphi)^2}} T - \dfrac{\kappa\sec\varphi}{\sqrt{\varphi'^2 + (\kappa\sec\varphi)^2}} N + \dfrac{\varphi'\cos\varphi}{\sqrt{\varphi'^2 + (\kappa\sec\varphi)^2}} B \\ B_2 = \dfrac{\kappa\tan\varphi}{\sqrt{\varphi'^2 + (\kappa\sec\varphi)^2}} T + \dfrac{\varphi'}{\sqrt{\varphi'^2 + (\kappa\sec\varphi)^2}} N + \dfrac{\kappa}{\sqrt{\varphi'^2 + (\kappa\sec\varphi)^2}} B \end{cases} \tag{1.16}$$

Corollary 3. *The Frenet vector fields of the involute curve α_2, can be written as the principal normal vector field on the curve α*

$$T_2 = \frac{N'}{\|N'\|},\ \ N_2 = \frac{\left(\frac{N'}{\|N'\|}\right)'}{\left\|\left(\frac{N'}{\|N'\|}\right)'\right\|},\ \ B_2 = T_2 \wedge N_2. \tag{1.17}$$

Corollary 4. *The first and the second curvatures of the second order involute α_2 of expression according to α are*

$$\begin{cases} \kappa_2 = \dfrac{\sqrt{\varphi'^2 + (\kappa\sec\varphi)^2}}{\lambda_1 \kappa\sec\varphi}, \lambda_1 = c_1 - \displaystyle\int \lambda\kappa ds \\ \tau_2 = \dfrac{\lambda}{\lambda_1 \sec\varphi}\left(\dfrac{\varphi'}{\kappa\sec\varphi}\right)'\left(\dfrac{\kappa\sec\varphi}{\sqrt{\varphi'^2 + (\kappa\sec\varphi)^2}}\right)^2 \end{cases} \tag{1.18}$$

Example. Let us consider the α curve, α_1 and α_2, respectively

$$\begin{aligned} \alpha(s) &= \Big(s\sin(s), s\cos(s), s^2\Big), \\ \alpha_1(s) &= \Big(2\sin(s) + 2s\cos(s) - s^2\cos(s), 2\cos(s) + 2s\sin(s) - s^2\sin(s), 4 - s^2\Big), \end{aligned}$$

$$
\begin{aligned}
\alpha_2(s) \quad = \quad & \Big(4\cos(s) + 11s^2\cos(s) - 16s\cos(s) - 2s\sin(s) - 6s^3\sin(s) + 9s^2\sin(s) \\
& -2s^3\cos(s) + s^4\sin(s) + 2\sin(s), -4\sin(s) - 11s^2\sin(s) + 16s\sin(s) \\
& -2s\cos(s) - 6s^3\cos(s) + 9s^2\cos(s) + 2s^3\sin(s) + s^4\cos(s) + 2\cos(s), \\
& 4 - 14s - 2s^3 + 11s^2\Big)
\end{aligned}
$$

where $c = 2$. In terms of definitions, Figure 3 follows

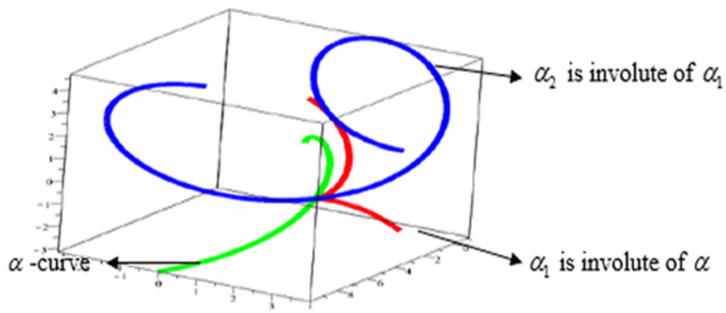

FIGURE 3. α, α_1 and α_2- curves

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