Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2018) 42: 1536 – 1541

doi:10.3906/mat-1612-45

On reflexivity of the Bochner space $L^p(\mu, E)$ for arbitrary μ

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Received: 13.12.2016 • **Accepted/Published Online:** 23.01.2018 • **Final Version:** 08.05.2018

Abstract: Let $(\Omega, \mathcal{A}, \mu)$ be a finite positive measure space, E a Banach space, and $1 < p < \infty$. It is known that the Bochner space $L^p(\mu, E)$ is reflexive if and only if E is reflexive. It is also known that $\mathcal{L}(L^1(\mu), E) = L^\infty(\mu, E)$ if and only if E has the Radon–Nikodým property. In this study, as an application of hyperstonean spaces, these results are extended to arbitrary measures by replacing the given measure space by an equivalent perfect one.

Key words: Bochner space, perfect measure, hyperstonean space

1. Introduction

Let $(\Omega, \mathcal{A}, \mu)$ be a positive measure space, E a Banach space, and $p \geq 1$ a real number. We shall denote the Bochner space $L^p(\Omega, \mathcal{A}, \mu, E)$ by $L^p(\mu, E)$ if there is no chance of ambiguity about the underlying measurable space. For definitions and properties of these spaces we refer to [4] or [5].

The space $L^\infty(\mu, E)$ is defined to be the space of all essentially bounded measurable functions from Ω to E. (Recall that a measurable function from $f : \Omega \longrightarrow E$ is essentially bounded if $\exists$ a number $\alpha > 0$ such that the set $\{x \in \Omega : \|f(x)\| > \alpha\}$ is locally null, i.e. its intersection with every set of finite measure has measure zero.)

A Banach space E is said to have the Radon–Nikodým property (RNP) with respect to a positive measure μ if every μ-continuous measure G from $\mathcal{A}$ to E that is of bounded variation can be represented by an integrable E-valued function g; that is,

$$G(A) = \int_\Omega g\chi_A d\mu = \int_A g d\mu \ \forall A \in \mathcal{A}.$$

We say that a Banach space E has the RNP if it has this property with respect to every positive measure. Many spaces, including reflexive ones (in particular, Hilbert spaces), and separable dual spaces, have the RNP.

Let us recall that a Banach space X is reflexive if the natural embedding $x \longrightarrow \widehat{x}$ from X into the second dual X^{**} is surjective, where for $x \in X$, $\widehat{x} : X^* \longrightarrow \mathbb{C}$ is defined by $\widehat{x}(\varphi) = \varphi(x)$, $\varphi \in X^*$.

Examples show that there are nonreflexive Banach spaces, which are linearly isometric to their second duals.

Let $(S, \mathcal{A}, \upsilon)$ be a *finite* positive measure space, E a Banach space, and $p > 1$ a *real* number.

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2010 *AMS Mathematics Subject Classification:* 46E40, 28B05, 47B38

(a) It is known that the Bochner space $L^p(\nu, E)$ is reflexive if and only if E is reflexive [4, p.100];

(b) it is also known that E has the RNP with respect to υ if and only if every bounded linear mapping $T : L^1(\upsilon) \longrightarrow E$ is representable, i.e. there is a function $g \in L^\infty(\nu, E)$ such that

$$Tf = \int_S fg d\mu, \; f \in L^1(\upsilon)$$

[4, p.63].

Without any serious difficulty, these results can be extended to σ-finite positive measures, but for arbitrary positive measures, it poses some serious difficulties, which we shall overcome by replacing the given measure space by an equivalent perfect one.

A Stonean space is a compact Hausdorff space in which the closure of every open set is open. These are precisely the Stonean spaces of complete Boolean algebras [9].

Definition 1 *A positive Borel measure μ on a Stonean space Ω is said to be perfect if:*
(i) every nonempty open set contains a clopen set with finite positive measure;
(ii) every nowhere dense Borel set has measure zero (equivalently, every closed set with empty interior has measure zero).

Definition 2 *A Stonean space Ω with a perfect measure μ on it is called hyperstonean, and the measure space $(\Omega, \mathcal{A}, \mu)$ is called a hyperstonean measure space.*

Definition 3 *Two measures (or measure spaces) are said to be equivalent if for each $1 \leq p < \infty$ the corresponding L^p spaces are linearly isometric.*

In [2], Cengiz proved that *every* positive measure space is equivalent to a hyperstonean measure space, which shows that the class of hyperstonean spaces is very large indeed, but not every Stonean space is hyperstonean; for instance, the Dedekind completion (order completion) of $C([0,1], \mathbb{R})$ is an order complete M-space, so order- and norm-isometric to $C(W, \mathbb{R})$ for some compact Hausdorff space W. The space W is known to be Stonean but not hyperstonean [4, p. 123]).

All perfect measures on the same Stonean space are equivalent to one another, and they are also absolutely continuous with respect to each other.

2. Main results

We shall fix a hyperstonean measure space $(\Omega, \mathcal{A}, \mu)$ and generalize the results (a) and (b) mentioned earlier. The measure μ has many nice properties, some of which will be mentioned in due course.

Now we are ready to prove the following theorem:

Theorem 4 *Let $(\Omega, \mathcal{A}, \mu)$ be a perfect measure space, E be a Banach space, and $1 < p < \infty$. Then $L^p(\mu, E)$ is reflexive if and only if E is reflexive.*

An application of Zorn's lemma provides us with a maximal family $\{\Omega_i : i \in I\}$ of disjoint clopen sets of strictly positive finite measure. Clearly, $\bigcup \Omega_i$ is open and dense in Ω, and since μ is perfect, $\Omega \setminus \bigcup \Omega_i$ has measure zero. Therefore, for any Banach space E and $1 \leq p < \infty$,

$$L^p(\Omega, \mathcal{A}, \mu) = \sum_i \oplus L^p(\Omega_i, \mu, E) \ \ (p\text{-direct sum}).$$

For the proof of the theorem, we need the following lemma.

Lemma 5 *Let $(\Omega, \mathcal{A}, \mu)$ and E be as in the theorem. If E has the Radon–Nikodým property with respect to each μ_i, $i \in I$ (the restriction of μ to the σ-algebra $\mathcal{A}_i = \{A \cap \Omega_i : A \in \mathcal{A}\}$), then it has the same property with respect to μ.*

Proof Let $G : \mathcal{A} \longrightarrow E$ be a μ-continuous vector measure of bounded variation. Then for each $i \in I$, there exists a Bochner integrable function $g_i : \Omega \longrightarrow E$ that vanishes outside Ω_i and such that

$$G(A) = \int_A g_i d\mu \text{ for all } A \in \mathcal{A}.$$

Since $|G|$ (the variation of G) is a finite measure on Ω, $|G|(\Omega_i) = 0$ for all but countably many i in I. Thus, we may assume that $I = \{1, 2, 3, \ldots\}$.

Now let $g = \sum_i g_i$. Then g is measurable and

$$|G|(\Omega) = \int_\Omega \|g(.)\| \, d\mu = \|g\|_1 < \infty,$$

that is, g is Bochner integrable.

Let $A \in \mathcal{A}$. Then, since

$$\left\| \int_A g d\mu - \sum_{i=1}^{n} \int_{A_i} g d\mu \right\| \leq \int_\Omega \chi_{B_n} \|g(.)\| \, d\mu$$

where $A_i = A \cap \Omega_i$ and $B_n = \bigcup_{i=n+1}^{\infty} A_n$, by the dominated convergence theorem,

$$\int_A g d\mu = \sum_{i=1}^{\infty} \int_{A_i} g d\mu = \sum_{i=1}^{\infty} G(A_i) = G(A),$$

proving our lemma. □

Proof of the theorem First let us assume that $L^p(\mu, E)$ is reflexive. Fix $i \in I$ and let χ_i denote the characteristic function of Ω_i. Then the mapping $u \longrightarrow \mu(\Omega_i)^{-1} u \chi_i$ is a linear isometry of E onto a closed subspace of $L^p(\mu, E)$. Since every closed subspace of a reflexive space is reflexive [8, p. 111], E is reflexive.

For the converse, we now assume that E is reflexive and identify it with its second dual E^{**}.

For $g \in L^q(\mu, E^*)$ we define ϕ_g on $L^p(\mu, E)$ by

$$\phi_g(f) = \int\limits_{\Omega} \langle f, g \rangle \, d\mu, \;\; f \in L^p(\mu, E),$$

where $\langle f, g \rangle (t) = g(t)(f(t)),\; t \in \Omega$. Notice that since $E^{**} = E,\; \langle g, f \rangle$ is also defined and equals $\langle f, g \rangle$.

We know that in the case in which μ is finite, the mapping $g \longrightarrow \phi_g$ maps $L^q(\mu, E^*)$ isometrically onto $L^p(\mu, E)^*$ if and only if E^* has the Radon–Nikodým property with respect to μ. (This theorem is due to [1] when μ is a Lebesgue measure on $[0,1]$. A nice proof of this result can be found in [4, pp. $97-100$].) In [3], Cengiz extended this theorem to arbitrary measures.

Being reflexive spaces, E and E^* have the Radon–Nikodým property with respect to finite measures [4, p. 76] and thus with respect to each $\mu_i,\; i \in I$, and therefore, by Lemma 5, they have this property with respect to μ. Consequently, by the above mentioned theorem, the map $g \longrightarrow \phi_g$ is a surjective linear isometry, and also by the same theorem, the map $h \longrightarrow \phi^*_h$ is a linear isometry from $L^p(\mu, E)$ onto $L^q(\mu, E^*)^*$, where for $h \in L^p(\mu, E)$, ϕ^*_h is defined on $L^q(\mu, E^*)$ similarly. Thus, $L^p(\mu, E)^* = \{\phi_g : g \in L^q(\mu, E^*)\}$ and $L^q(\mu, E^*)^* = \{\phi^*_h : h \in L^p(\mu, E)\}$.

Now let $\psi \in L^p(\mu, E)^{**}$. Then the mapping $g \longrightarrow \psi(\phi_g),\; g \in L^q(\mu, E^*)$, is a bounded functional on $L^q(\mu, E^*)$. Therefore, there exists an $h \in L^p(\mu, E)$ such that $\psi = \phi^*_h$; that is, for each $g \in L^q(\mu, E^*)$ we have

$$\begin{aligned} \psi(\phi_g) &= \phi^*_h(g) = \int\limits_{\Omega} \langle g, h \rangle \; d\mu = \int \langle g, h \rangle \; d\mu \\ &= \phi_g(h) = \widehat{h}(\phi_g), \end{aligned}$$

where $\widehat{h}$ denotes the image of h in $L^p(\mu, E)^{**}$ under the natural embedding. This shows that $\psi = \widehat{h}$, and since ψ was arbitrary we conclude that $L^p(\mu, E)$ is reflexive. Hence, the proof is completed. □

Remark 6 *One may argue that*

$$L^p(\mu, E)^{**} \simeq L^q(\mu, E^*)^* \simeq L^p(\mu, E^{**}) \simeq L^p(\mu, E),$$

that is, $L^p(\mu, E)^{**}$ is isometric to $L^p(\mu, E)$, but this does NOT imply that $L^p(\mu, E)$ is reflexive, for there are nonreflexive spaces that are isometric to their second duals.

Finally we extend result (b) mentioned in the introductory part to arbitrary measures.

Theorem 7 *Let $(\Omega, \mathcal{A}, \mu)$ be an arbitrary perfect measure space and E be a Banach space. Then E has the Radon–Nikodým property with respect to μ if and only if every bounded linear operator T from $L^1(\mu)$ to E is representable, i.e. $\mathcal{L}(L^1(\mu), E) = L^\infty(\mu, E)$, where $\mathcal{L}(L^1(\mu), E)$ denotes the space of all bounded linear operators from $L^1(\mu)$ to E.*

Proof Assume that each operator in $\mathcal{L}(L^1(\mu), E)$ is representable. Fix $i \in I$ and let $T_i : L^1(\Omega_i) \longrightarrow E$ be a bounded linear operator, where $L^1(\Omega_i)$ denotes the subspace of the elements of $L^1(\mu)$ vanishing outside Ω_i. We extend T_i to all of $L^1(\mu)$ by

$$T(f) = T_i(f\chi_i), \;\; f \in L^1(\mu),$$

where χ_i denotes the characteristic function of Ω_i. Then, by our assumption, there exists a $g \in L^{\infty}(\mu, E)$ such that

$$T(f) = \int_{\Omega} fg d\mu \text{ for all } f \in L^1(\mu).$$

For each measurable set A of finite measure contained in the complement of Ω_i we have

$$0 = T_i(\chi_A \chi_i) = T(\chi_A) = \int_{\Omega} \chi_A g d\mu = \int_{A} g d\mu.$$

Thus, it follows that $g(x) = 0$ a.e. outside Ω_i, i.e. $g \in L^{\infty}(\Omega_i, E)$, and T_i is represented by g.

The above discussion proves that E has the Radon–Nikodým property with respect to $\mu_i = \mu \mid_{\Omega_i}$, and hence, by Lemma 5, it has this property with respect to μ.

Conversely, assume that E has the Radon–Nikodým property with respect to μ and let $T : L^1(\mu) \longrightarrow E$ be a bounded linear operator. For each $i \in I$, let T_i denote the restriction of T to $L^1(\Omega_i)$. Then, by the finite measure case, T_i is representable. Let, for $i \in I$, $g_i \in L^{\infty}(\Omega_i, E)$ such that

$$T_i(f) = \int_{\Omega_i} fg_i d\mu \text{ for all } f \in L^1(\Omega_i).$$

Now let $g = \sum_i g_i$. Then g is locally measurable and since $\|T_i\| = \|g_i\|_{\infty} \leq \|T\|$ for each $i \in I$, we have $\|g\|_{\infty} \leq \|T\|$. It remains to show that, for $f \in L^1(\mu)$,

$$T(f) = \int_{\Omega} fg d\mu.$$

Let $f \in L^1(\mu)$. Since its support is σ-finite, in the rest of the proof we will assume that $I = \{1, 2, 3, \ldots\}$. Let $A_n = \bigcup_{i=1}^{n} \Omega_i$. Since, by the dominated convergence theorem, $f\chi_{A_n} \underset{n}{\longrightarrow} f$ in $L^1(\mu)$ and T is continuous, $T(f\chi_{A_n}) \underset{n}{\longrightarrow} T(f)$, and since

$$\begin{aligned} \left\| \int_{\Omega} fg d\mu - T(f\chi_{A_n}) \right\| &= \left\| \int_{\Omega} fg d\mu - \int_{\Omega} \chi_{A_n} fg d\mu \right\| \\ &\leq \|g\|_{\infty} \int_{\Omega} \chi_{A_n'} \|f(.)\| \, d\mu \end{aligned}$$

where $A_n^{'}$ denotes the complement of A_n, again by the dominated convergence theorem we obtain

$$T(f) = \lim_n T(f\chi_{A_n}) = \int_{\Omega} fg d\mu.$$

$\square$

Remark 8 *This last result need not hold even in the scalar case if the measure space is not decomposable, for there are measures* ν *such that* $L^1(\nu)^* \neq L^\infty(\nu)$ *(see* [6, *p.* 349] *or* [7, *p.* 287]*).*

Remark 9 *We would not have been able to obtain the extension results in the theorems if it had not been for the equivalence of any arbitrary positive measure to a perfect one.*

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